

# ON THE LAPLACE TRANSFORM FOR TEMPERATE HOLOMORPHIC FUNCTIONS

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ABSTRACT. In order to discuss the Fourier-Sato transform of not necessarily conic sheaves, we compensate the lack of homogeneity by adding an extra variable. We can then obtain Paley-Wiener type results, using a theorem by Kashiwara and Schapira on the Laplace transform for temperate holomorphic functions. As a key tool in our approach, we introduce the subanalytic sheaf of holomorphic functions with exponential growth, which should be of independent interest.

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## INTRODUCTION

Let  $\mathbb{V}$  and  $\mathbb{V}^*$  be dual  $n$ -dimensional complex vector spaces. Kashiwara-Schapira [11] proved that the Laplace transform

$$\varphi(x) \mapsto \int e^{-\langle x, y \rangle} \varphi(x) dx$$

induces an isomorphism

$$(0.1) \quad \mathrm{RHom}(F, \mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) \simeq \mathrm{RHom}(F^\wedge[n], \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^t).$$

Here,  $F$  is a conic  $\mathbb{R}$ -constructible complex of sheaves on  $\mathbb{V}$ ,  $F^\wedge$  its Fourier-Sato transform, and  $\mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t$  is the subanalytic sheaf of holomorphic functions tempered up to the projective compactification  $\mathbb{P}(\mathbb{V})$  of  $\mathbb{V}$ . Let us recall a couple of statements deduced from (0.1) for particular choices of  $F$ .

(i) Assume that  $\mathbb{V}$  and  $\mathbb{V}^*$  are complexifications of the real vector spaces  $\mathbf{V}$  and  $\mathbf{V}^*$ . For  $F = \mathbf{k}_{\mathbf{V}}$  the constant sheaf on  $\mathbf{V}$ , one recovers the classical isomorphism

$$\Gamma(\mathbf{V}; \mathcal{D}b_{\mathbf{V}|\mathbb{P}(\mathbf{V})}^t) \simeq \Gamma(\mathbf{V}^*; \mathcal{D}b_{\mathbf{V}^*|\mathbb{P}(\mathbf{V}^*)}^t)$$

between the spaces of tempered distributions.

(ii) Let  $(x', x'')$  be the coordinates on  $\mathbb{V} = \mathbb{C}^p \times \mathbb{C}^q$  and  $(y', y'')$  the dual coordinates on  $\mathbb{V}^*$ . Let  $A = \{(\mathrm{Re}x')^2 - (\mathrm{Re}x'')^2 \leq 0\}$  be a quadratic cone in  $\mathbf{V}$ . For  $F = \mathbf{k}_A$ , one recovers a result of Faraut-Gindikin:

$$\Gamma_A(\mathbf{V}; \mathcal{D}b_{\mathbf{V}|\mathbb{P}(\mathbf{V})}^t) \simeq H^q \mathrm{R}\Gamma_{\{(\mathrm{Re}y')^2 - (\mathrm{Re}y'')^2 \geq 0\}}(\mathbb{V}^*, \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^t).$$

Our aim in this paper is to extend the isomorphism (0.1) in order to treat the case where  $F$  is not necessarily conic. This will allow us to obtain Paley-Wiener type results like the following:

(iii) Let  $A \subset \mathbf{V}$  be a closed, convex, subanalytic, bounded subset. Denote by  $h_A$  its support function. For  $F = \mathbf{k}_A$  we will recover the classical Paley-Wiener theorem of [7, Theorem 7.3.1]:

$$\Gamma_A(\mathbf{V}; \mathcal{D}b_{\mathbf{V}}) \xrightarrow{\sim} \{\psi \in \Gamma(\mathbb{V}^*; \mathcal{O}_{\mathbb{V}^*}) : \exists c, \exists m, \forall y, |\psi(y)| \leq c(1 + |y|)^m e^{h_A(-\mathrm{Re}y)}\}.$$

We will also discuss the case where  $A$  is not necessarily bounded nor included in the real part  $\mathbf{V}$  of  $\mathbb{V}$ .

(iv) Generalizing (ii) above, for  $c \geq 0$  let  $A = \{(\mathrm{Re}x')^2 - (\mathrm{Re}x'')^2 \leq c^2\}$  be a quadric in  $\mathbf{V}$ . For  $F = \mathbf{k}_A$  we will get a description of the Laplace transform of the space  $\Gamma_A(\mathbf{V}; \mathcal{D}b_{\mathbf{V}|\mathbb{P}(\mathbf{V})}^t)$ .

In order to state our result, let us start by describing the functional spaces that will appear in the statement.

Let  $j: X \rightarrow X'$  be an open subanalytic embedding of real analytic manifolds. A  $j$ - $\mathbb{R}$ -constructible sheaf on  $X$  is a sheaf (or more precisely,

an object of the derived category of sheaves) whose proper direct image by  $j$  is  $\mathbb{R}$ -constructible in  $X'$ . Such sheaves are naturally identified with sheaves on the site  $X_{j\text{-sa}}$ , whose objects are open subsets of  $X$  which are subanalytic in  $X'$  and whose coverings are locally finite in  $X'$ .

For  $f: X \rightarrow \mathbb{R}$  a continuous subanalytic function, consider the sheaf  $\mathcal{C}_{X|X'}^{\infty,[f]}$  on  $X_{j\text{-sa}}$  whose sections on  $U \subset X$  are  $f$ -tempered functions. These are smooth functions  $\varphi$  which, together with all of their derivatives, locally satisfy on  $X'$  an estimate of the type

$$|\varphi(x)| \leq c \left( 1 + \frac{1}{\text{dist}(X \setminus U, x)} + |f(x)| \right)^m e^{f(x)}.$$

The subanalytic sheaf of tempered functions considered in [12] is recovered as  $\mathcal{C}_X^{\infty,t} = \mathcal{C}_{X|X'}^{\infty,[0]}$ . The sheaf  $\mathcal{C}_{X|X'}^{\infty,t} = \mathcal{C}_{X|X'}^{\infty,[0]}$  takes also into account growth conditions at infinity. We show that  $\varphi$  is  $f$ -tempered on  $U$  if and only if  $\varphi(x)e^s$  is tempered on  $\{(x, s): x \in U, s < -f(x)\}$ .

Let now  $j: X \rightarrow X'$  be an open subanalytic embedding of complex analytic manifolds. We denote by  $\mathcal{O}_{X|X'}^{[f]}$  the Dolbeault complex with coefficients in  $\mathcal{C}_{X|X'}^{\infty,[f]}$ . These sheaves should be of independent interest in dealing with holonomic  $\mathcal{D}$ -modules which are not necessarily regular.

The functional spaces we will be dealing with are those of the form

$$\text{RHom}(F, \mathcal{O}_{X|X'}^{[f]}),$$

for  $F$  a  $j$ - $\mathbb{R}$ -constructible sheaf. For example, the subanalytic sheaf of holomorphic functions tempered up to infinity appearing in (0.1) is recovered as  $\mathcal{O}_{X|X'}^t = \mathcal{O}_{X|X'}^{[0]}$ .

We can now state our result on the Laplace transform. Recall that  $\mathbb{V}$  and  $\mathbb{V}^*$  are dual complex vector spaces of dimension  $n$ . In order to treat the case of not necessarily conic sheaves, we compensate the lack of homogeneity by adding an extra variable. Consider the embedding

$$i: \mathbb{V} \rightarrow \mathbb{V} \times \mathbb{C}, \quad x \mapsto (x, -1).$$

Let  $F$  be a  $j$ - $\mathbb{R}$ -constructible sheaf on  $\mathbb{V}$ , where  $j: \mathbb{V} \rightarrow \mathbb{P}(\mathbb{V})$  is the complex projective compactification. Note that if  $F$  is conic, then

$$(Ri_! F)^\wedge \simeq (F^\wedge \boxtimes \mathbf{k}_{\mathbb{C}}) \otimes \mathbf{k}_{\{\text{Ret} \geq 0\}},$$

with  $t \in \mathbb{C}$  the dual of the extra variable. For  $F$  not necessarily conic, assume that

$$(Ri_! F)^\wedge \simeq (G \boxtimes \mathbf{k}_{\mathbb{C}}) \otimes \mathbf{k}_{\{\text{Ret} \geq -g(y)\}},$$

for  $G$  a conic sheaf on  $\mathbb{V}^*$  and  $g: \mathbb{V}^* \rightarrow \mathbb{R}$  a continuous subanalytic function which is homogeneous of degree one. (We state in Appendix A a conjecture suggesting that this assumption is not very strong.) Then we have an isomorphism

$$\text{RHom}(F, \mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) \simeq \text{RHom}(G[n], \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^{[g]}).$$

In order to get this result we will start by discussing some generalities on conic sheaves. For example, we explicitly describe the left and right adjoint to the embedding of conic sheaves into sheaves. We also prove that the functor  $(Ri_!(\cdot))^\wedge$  is fully faithful and that its essential image consists of the conic sheaves  $H$  on  $\mathbb{V}^* \times \mathbb{C}$  such that

$$H * \mathbf{k}_{\{\mathrm{Re}t \geq 0, y=0\}} \xrightarrow{\sim} H.$$

This is the kind of condition considered by Tamarkin [17]. We will see how the Fourier transform considered in [17] is related to the functor  $F \mapsto (Ri_!F)^\wedge$ .

The plan of the paper is as follows.

Section 1 recalls the formalism of kernel calculus for sheaves, which will be useful on several occasions in the rest of the paper.

On a space  $X$  endowed with an  $\mathbb{R}^+$ -action, Section 2 gives an elementary construction of the left and right adjoint to the embedding of conic sheaves into sheaves. These are called conification functors.

Let  $i: Y \rightarrow X$  be the embedding of a locally closed subset satisfying a suitable assumption with respect to the  $\mathbb{R}^+$ -action. Section 3 characterizes the image of the fully faithful functor sending a not necessarily conic sheaf on  $Y$  to the conification of its direct image by  $i$ .

Section 4 recalls some properties of the Fourier-Sato transform between conic sheaves on dual real vector spaces  $\mathbb{V}$  and  $\mathbb{V}^*$ .

Section 5 characterizes the image of the fully faithful functor from  $\mathbb{V}$  to  $\mathbb{V}^* \times \mathbb{R}$  sending a not necessarily conic sheaf on  $\mathbb{V}$  to the Fourier-Sato transform of its direct image by the embedding  $i: \mathbb{V} \rightarrow \mathbb{V} \times \mathbb{R}$ ,  $x \mapsto (x, -1)$ .

Let  $j: X \rightarrow X'$  be an open subanalytic embedding of real analytic manifolds. The  $j$ -subanalytic site on  $X$  is the one induced by the subanalytic site on  $X'$ . In Section 6 we introduce the  $j$ -subanalytic sheaf of smooth functions with exponential growth. We also relate this sheaf to the sheaf of tempered smooth functions with an extra variable.

If  $X$  is a complex analytic manifold, we discuss in Section 7 the  $j$ -subanalytic sheaf of holomorphic functions with exponential growth. This is the Dolbeault complex of the previous sheaf.

In Section 8 we recall a theorem by Kashiwara and Schapira on the Fourier-Laplace transform between temperate holomorphic functions associated with conic sheaves on dual complex vector spaces  $\mathbb{V}$  and  $\mathbb{V}^*$ .

We then extend it in Section 9 to sheaves which are not necessarily conic by considering as above the embedding  $i: \mathbb{V} \rightarrow \mathbb{V} \times \mathbb{C}$ ,  $x \mapsto (x, -1)$ .

As an application of the above results, we get in Section 10 some Paley-Wiener type theorems.

In Appendix A we show how the functor  $F \mapsto (Ri_!F)^\wedge$  is expressed in terms of the Fourier transform considered by Tamarkin in [17].

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## 1. KERNEL CALCULUS

We recall here the definition and basic properties of kernel calculus for sheaves. This formalism will be useful in the first part of the paper.

Let  $X$  be a locally compact topological space and  $\mathbf{k}$  a field. For  $A \subset X$  a locally closed subset, we denote by  $\mathbf{k}_A$  the constant sheaf on  $A$  with stalk  $\mathbf{k}$ , extended by zero to  $X$ . Denote by  $\mathbf{D}^b(\mathbf{k}_X)$  the bounded derived category of sheaves of  $\mathbf{k}$ -vector spaces on  $X$  and by  $\otimes$ ,  $R\mathcal{H}om$ ,  $f^{-1}$ ,  $Rf_*$ ,  $Rf_!$ ,  $f^!$  the usual operations (here  $f: X \rightarrow Y$  is a continuous map of locally compact spaces). More generally, in this paper we will follow the notations of [9].

Let  $X_i$  ( $i \in \mathbb{N}$ ) be locally compact topological spaces. Consider the projections  $q_{ij}: X_1 \times X_2 \times X_3 \rightarrow X_i \times X_j$ . For  $K_{ij} \in \mathbf{D}^b(\mathbf{k}_{X_i \times X_j})$ , set

$$(1.1) \quad K_{12} \circ_{X_2} K_{23} = Rq_{13!}(q_{12}^{-1}K_{12} \otimes q_{23}^{-1}K_{23}),$$

$$(1.2) \quad [K_{12}, K_{23}]_{X_2} = Rq_{13*}R\mathcal{H}om(q_{12}^{-1}K_{12}, q_{23}^!K_{23}).$$

By adjunction and projection formula, one gets

$$(1.3) \quad [K_{12} \circ_{X_2} K_{23}, K_{34}]_{X_3} \simeq [K_{12}, [K_{23}, K_{34}]_{X_3}]_{X_2}.$$

The operations (1.1) and (1.2) are called compositions of kernels. This is because, for  $K \in \mathbf{D}^b(\mathbf{k}_{X \times Y})$ , the functors

$$(1.4) \quad \mathbf{D}^b(\mathbf{k}_X) \rightarrow \mathbf{D}^b(\mathbf{k}_Y), \quad F \mapsto F \circ_X K,$$

$$(1.5) \quad \mathbf{D}^b(\mathbf{k}_Y) \rightarrow \mathbf{D}^b(\mathbf{k}_X), \quad G \mapsto [K, G]_Y,$$

can be considered as sheaf theoretical analogues of an integral transform with kernel  $K$ . Note that (1.3) implies that (1.4) and (1.5) are adjoint functors.

Denote by  $\Gamma_f \subset X \times Y$  the graph of  $f: X \rightarrow Y$  and by  $\boxtimes$  the exterior tensor product. One has

$$F \boxtimes G \simeq F \circ_{\{pt\}} G, \quad Rf_!F \simeq F \circ_X \mathbf{k}_{\Gamma_f}, \quad f^{-1}G \simeq \mathbf{k}_{\Gamma_f} \circ_Y G.$$

Similar relations hold for the adjoint operations.

Denote by  $\Delta_{X_i}$  the diagonal of  $X_i \times X_i$  and set  $K_{ij}^r = Rr_! K_{ij}$ , for  $r(x_i, x_j) = (x_j, x_i)$ . One has

$$(1.6) \quad \mathbf{k}_{\Delta_{X_1}} \circ_{X_1} K_{12} \simeq K_{12} \simeq [\mathbf{k}_{\Delta_{X_1}}, K_{12}]_{X_1},$$

$$(1.7) \quad (K_{12} \circ_{X_2} K_{23}) \circ_{X_3} K_{34} \simeq K_{12} \circ_{X_2} (K_{23} \circ_{X_3} K_{34}),$$

$$(1.8) \quad (K_{12} \circ_{X_2} K_{23})^r \simeq K_{23}^r \circ_{X_2} K_{12}^r.$$

Consider the projections  $q_{ij}: X_1 \times \cdots \times X_{m+1} \rightarrow X_i \times X_j$ . One has

$$(1.9) \quad K_{12} \circ_{X_2} \cdots \circ_{X_m} K_{mm+1} \simeq Rq_{1m+1!}(q_{12}^{-1} K_{12} \otimes \cdots \otimes q_{mm+1}^{-1} K_{mm+1}).$$

## 2. CONIC SHEAVES

Let  $X$  be a locally compact space endowed with an action of the multiplicative group  $\mathbb{R}^+$  of positive real numbers and consider the maps

$$\mathbb{R}^+ \times X \xrightleftharpoons[\mu]{p} X,$$

where  $p$  is the projection and  $\mu$  is the action. We will write for short  $\mu(t, x) = tx$ . Recall (see [9, §3.7]) that an object  $F \in \mathbf{D}^b(\mathbf{k}_X)$  is called  $\mathbb{R}^+$ -conic if it satisfies

$$\mu^{-1} F \simeq p^{-1} F.$$

Note that  $p^! \simeq p^{-1}[1]$  and  $\mu^! \simeq \mu^{-1}[1]$ . Denote by  $\mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_X)$  the full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_X)$  whose objects are  $\mathbb{R}^+$ -conic.

**Definition 2.1.** The left and right conification functors are the pair of adjoint functors

$$\begin{aligned} (\cdot)^c: \mathbf{D}^b(\mathbf{k}_X) &\rightarrow \mathbf{D}^b(\mathbf{k}_X), & F^c &= R\mu_! p^{-1} F[1] \simeq R\mu_! p^! F, \\ {}^c(\cdot): \mathbf{D}^b(\mathbf{k}_X) &\rightarrow \mathbf{D}^b(\mathbf{k}_X), & {}^c F &= Rp_* \mu^! F[-1] \simeq Rp_* \mu^{-1} F. \end{aligned}$$

Note that  $\mu$  and  $p$  can be interchanged in the above definition. In fact, one has for example  $R\mu_! p^{-1} \simeq R\mu_! Re_! e^{-1} p^{-1} \simeq Rp_! \mu^{-1}$ , where  $e(t, x) = (t^{-1}, tx)$ .

In this section we will show that the left and right conification functors are respectively the left and right adjoint to the embedding  $\mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_X) \rightarrow \mathbf{D}^b(\mathbf{k}_X)$ .

**Remark 2.2.** Let  $j: X \rightarrow \mathbb{R}^+ \times X$  be the embedding  $x \mapsto (1, x)$ . Assume that there is an isomorphism  $\mu^{-1} F \simeq p^{-1} F$ . Since  $p^{-1}$  is fully faithful, one checks that there is a unique isomorphism  $\beta: \mu^{-1} F \xrightarrow{\sim} p^{-1} F$  such that  $j^{-1} \beta = \text{id}_F$ . Thus, the category  $\mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_X)$  is equivalent to the equivariant derived category in the sense of [2]. There, for groups  $G$  more general than  $\mathbb{R}^+$ , it is shown that the forgetful functor  $\mathbf{D}_G^b(\mathbf{k}_X) \rightarrow \mathbf{D}^b(\mathbf{k}_X)$  has a left and a right adjoint. Here, for  $G = \mathbb{R}^+$ , we describe these adjoints without the machinery of equivariant derived categories.

Consider the projection  $q_{23}: \mathbb{R}^+ \times X \times X \rightarrow X \times X$ . Denoting by  $(t, x, x')$  a point in  $\mathbb{R}^+ \times X \times X$ , we will sometimes write  $\{x' = tx\}$  instead of  $\Gamma_\mu$ .

**Definition 2.3.** The conification kernel is the object of  $\mathbf{D}^b(\mathbf{k}_{X \times X})$  given by

$$C_X = Rq_{23!} \mathbf{k}_{\{x'=tx\}}[1].$$

**Lemma 2.4.** *There are isomorphisms:*

$$(2.1) \quad C_X \simeq (C_X)^r,$$

$$(2.2) \quad C_X \simeq \mathbf{k}_{r(\Gamma_\mu)} \circ_{\mathbb{R}^+ \times X} \mathbf{k}_{\Gamma_p}[1], \quad C_X \simeq \mathbf{k}_{r(\Gamma_p)} \circ_{\mathbb{R}^+ \times X} \mathbf{k}_{\Gamma_\mu}[1],$$

$$(2.3) \quad C_X \circ_X \mathbf{k}_{r(\Gamma_\mu)} \simeq C_X \circ_X \mathbf{k}_{r(\Gamma_p)}, \quad \mathbf{k}_{\Gamma_\mu} \circ_X C_X \simeq \mathbf{k}_{\Gamma_p} \circ_X C_X.$$

*Proof.* Set  $e(t, x, x') = (t^{-1}, x', x)$ . Then  $q_{23}e = rq_{23}$  and  $e(\{x' = tx\}) = \{x' = tx\}$ . Hence

$$Rq_{23!} \mathbf{k}_{\{x'=tx\}} \simeq Rq_{23!} Re! \mathbf{k}_{\{x'=tx\}} \simeq Rr! Rq_{23!} \mathbf{k}_{\{x'=tx\}}.$$

This proves (2.1).

Consider the projection  $q: X \times (\mathbb{R}^+ \times X) \times X \rightarrow \mathbb{R}^+ \times X \times X$ ,  $(x, t, \tilde{x}, x') \mapsto (t, x, x')$  and set  $q'_{14} = q_{23}q$ . One has

$$\begin{aligned} \mathbf{k}_{r(\Gamma_\mu)} \circ_{\mathbb{R}^+ \times X} \mathbf{k}_{\Gamma_p} &= Rq'_{14!} \mathbf{k}_{\{x=t\tilde{x}, x'=\tilde{x}\}} \\ &\simeq Rq_{23!} Rq! \mathbf{k}_{\{x=t\tilde{x}, x'=\tilde{x}\}} \simeq Rq_{23!} \mathbf{k}_{\{x'=tx\}}. \end{aligned}$$

This proves the first isomorphism in (2.2). The second one follows using (1.8) and (2.1).

Consider the projection  $q: \mathbb{R}^+ \times X \times X \times X \times \mathbb{R}^+ \rightarrow X \times X \times \mathbb{R}^+$ ,  $(t, x, x', x'', t') \mapsto (x, x'', t')$ . Consider the subsets of the source space

$$\begin{aligned} M &= \{(t, x, x', x'', t') : x' = tx, x' = t'x''\}, \\ P &= \{(t, x, x', x'', t') : x' = tx, x' = x''\}. \end{aligned}$$

By (1.9) one has

$$C_X \circ_X \mathbf{k}_{r(\Gamma_\mu)} \simeq Rq! \mathbf{k}_M[1], \quad C_X \circ_X \mathbf{k}_{r(\Gamma_p)} \simeq Rq! \mathbf{k}_P[1].$$

Let  $e(t, x, x', x'', t') = (tt', x, t'x', x'', t')$ . Since  $q = qe$  and  $e(P) = M$ , one has

$$Rq! \mathbf{k}_P \simeq Rq! Re! \mathbf{k}_P \simeq Rq! \mathbf{k}_{e(P)} \simeq Rq! \mathbf{k}_M.$$

This proves the first isomorphism in (2.3). The second one follows using (1.8) and (2.1).  $\square$

Note that for  $F \in \mathbf{D}^b(\mathbf{k}_X)$  there is a natural morphism

$$(2.4) \quad \alpha: F \rightarrow F^c,$$

defined as follows. Consider the embedding  $j: X \rightarrow \mathbb{R}^+ \times X$ ,  $x \mapsto (1, x)$ . Then  $\alpha$  is given by the composite

$$F \simeq R\mu_! Rj_! j^! p^! F \rightarrow R\mu_! p^! F \simeq F^c.$$

In terms of kernels, this is induced by the morphism

$$\mathbf{k}_{\{t=1, x'=tx\}} \xrightarrow{\sim} R\Gamma_{\{t=1\}} \mathbf{k}_{\{x'=tx\}}[1] \rightarrow \mathbf{k}_{\{x'=tx\}}[1],$$

noticing that  $Rq_{23!} \mathbf{k}_{\{t=1, x'=tx\}} \simeq \mathbf{k}_{\Delta_X}$ .

**Proposition 2.5.** *Let  $F \in \mathbf{D}^b(\mathbf{k}_X)$ .*

- (i) *There are isomorphisms  $F^c \simeq F \circ_X C_X \simeq C_X \circ_X F$ .*
- (ii)  *$F^c$  is  $\mathbb{R}^+$ -conic.*
- (iii)  *$F$  is  $\mathbb{R}^+$ -conic if and only if the morphism  $\alpha$  in (2.4) is an isomorphism.*
- (iv) *The functor  $(\cdot)^c$  is left adjoint to the fully faithful embedding  $\mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_X) \rightarrow \mathbf{D}^b(\mathbf{k}_X)$ . In particular,  $F^{cc} \simeq F^c$ .*
- (v) *Similar results hold for the right conification  ${}^c F \simeq [C_X, F]_X$ .*

*Proof.* (i) The isomorphism  $F \circ_X C_X \simeq C_X \circ_X F$  follows from (1.8) and (2.1). The isomorphism  $F^c \simeq C_X \circ_X F$  follows from (2.2).

(ii) We have to prove that  $\mu^{-1}(F^c) \simeq p^{-1}(F^c)$ . This is equivalent to

$$F \circ_X C_X \circ_X \mathbf{k}_{r(\Gamma_\mu)} \simeq F \circ_X C_X \circ_X \mathbf{k}_{r(\Gamma_p)},$$

which follows from (2.3).

(iii) If  $\alpha$  is an isomorphism, then  $F$  is  $\mathbb{R}^+$ -conic by (ii). If  $F$  is  $\mathbb{R}^+$ -conic, then  $F^c = Rp_! \mu^{-1} F[1] \simeq Rp_! p^{-1} F[1] \simeq F$ . Here, the last isomorphism follows from the isomorphism  $\mathbf{k}_{r(\Gamma_p)} \circ_{\mathbb{R}^+ \times X} \mathbf{k}_{\Gamma_p} \simeq \mathbf{k}_{\Delta_X}[-1]$ .

(iv) Let  $F \in \mathbf{D}^b(\mathbf{k}_X)$  and  $H \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_X)$ . By the analogue of (iii) for the right conification functor, one has  $H \simeq {}^c H$ . Then

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}^b(\mathbf{k}_X)}(F, H) &\simeq \mathrm{Hom}_{\mathbf{D}^b(\mathbf{k}_X)}(F, {}^c H) \\ &\simeq \mathrm{Hom}_{\mathbf{D}^b(\mathbf{k}_X)}(F^c, H) \\ &\simeq \mathrm{Hom}_{\mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_X)}(F^c, H). \end{aligned}$$

(v) The similar results for the right conification functor can be obtained by adjunction. For example, let us show that  ${}^c F$  is  $\mathbb{R}^+$ -conic. For any  $G \in \mathbf{D}^b(\mathbf{k}_{\mathbb{R}^+ \times X})$  one has

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}^b(\mathbf{k}_{\mathbb{R}^+ \times X})}(G, \mu^!({}^c F)) &\simeq \mathrm{Hom}_{\mathbf{D}^b(\mathbf{k}_X)}((R\mu_! G)^c, F) \\ &\simeq \mathrm{Hom}_{\mathbf{D}^b(\mathbf{k}_X)}((Rp_! G)^c, F) \\ &\simeq \mathrm{Hom}_{\mathbf{D}^b(\mathbf{k}_{\mathbb{R}^+ \times X})}(G, p^!({}^c F)), \end{aligned}$$



where the second isomorphism follows from (2.3). Hence  $\mu^!(^cF) \simeq p^!(^cF)$ .  $\square$

### 3. CONIFIED SHEAVES

Let  $X$  be a locally compact topological space endowed with an  $\mathbb{R}^+$ -action.

**Notation 3.1.** Let  $S \subset X$  be a locally closed subset.

- (i) Denote by  $\mathbf{D}_{\langle S \rangle}^b(\mathbf{k}_X)$  the full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_X)$  whose objects  $F$  satisfy  $F_{X \setminus S} = 0$ .
- (ii) Denote by  $\mathbf{D}_S^b(\mathbf{k}_X)$  the full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_X)$  whose objects  $F$  satisfy  $R\Gamma_{X \setminus S}F = 0$ .

**Definition 3.2.** Let us say that a subset  $Y \subset X$  is  $\mathbb{R}^+$ -simple if it is locally closed and if the multiplication  $\mu$  induces a topological isomorphism between  $\mathbb{R}^+ \times Y$  and the orbit  $\mathbb{R}^+Y \subset X$  endowed with the induced topology.

**Lemma 3.3.** *If  $Y$  is  $\mathbb{R}^+$ -simple, then  $\mathbb{R}^+Y$  is locally closed in  $X$ .*

*Proof.* Since  $Y$  is locally closed, it is locally compact. Then  $\mathbb{R}^+ \times Y$  is locally compact, and thus so is  $\mathbb{R}^+Y$  for the induced topology. It follows that  $\mathbb{R}^+Y$  is locally closed in  $X$ .  $\square$

**Proposition 3.4.** *Let  $i: Y \rightarrow X$  be the embedding of an  $\mathbb{R}^+$ -simple subset. There are equivalences*

$$\begin{aligned} \mathbf{D}^b(\mathbf{k}_Y) &\xrightarrow{\sim} \mathbf{D}_{\mathbb{R}^+, \langle \mathbb{R}^+Y \rangle}^b(\mathbf{k}_X), & G &\mapsto (Ri_!G)^c, \\ \mathbf{D}^b(\mathbf{k}_Y) &\xrightarrow{\sim} \mathbf{D}_{\mathbb{R}^+, \mathbb{R}^+Y}^b(\mathbf{k}_X), & G &\mapsto {}^c(Ri_*G), \end{aligned}$$

with quasi inverses  $i^{-1}[-1]$  and  $i^![1]$ , respectively.

*Proof.* (i) For the first equivalence, it is enough to prove that for  $G \in \mathbf{D}^b(\mathbf{k}_Y)$  and  $F \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_X)$  there are isomorphisms

$$(3.1) \quad i^{-1}((Ri_!G)^c) \simeq G[1], \quad (Ri_!i^{-1}F)^c \simeq F_{\mathbb{R}^+Y}[1].$$

In fact, (3.1) proves in particular that  $(Ri_!G)^c \in \mathbf{D}_{\langle \mathbb{R}^+Y \rangle}^b(\mathbf{k}_X)$ , since one has

$$(Ri_!G)^c \simeq (Ri_!i^{-1}((Ri_!G)^c))^c[-1] \simeq ((Ri_!G)^c)_{\mathbb{R}^+Y}.$$

(i-a) One has

$$i^{-1}((Ri_!G)^c) \simeq G \circ_Y \mathbf{k}_{\Gamma_i} \circ_X C_X \circ_X \mathbf{k}_{r(\Gamma_i)}.$$

Consider the projection  $q: Y \times X \times \mathbb{R}^+ \times X \times Y \rightarrow Y \times Y$  given by  $q(y, x, t, x', y') = (y, y')$  and the subset

$$Q = \{(y, x, t, x', y') : x = i(y), x' = tx, x' = i(y')\}$$

of the source space. Since  $Y$  is  $\mathbb{R}^+$ -simple, the equality  $i(y') = ti(y)$  implies  $t = 1$ . Hence

$$\mathbf{k}_{\Gamma_i} \circ_X C_X \circ_X \mathbf{k}_{r(\Gamma_i)} \simeq Rq_! \mathbf{k}_Q[1] \simeq \mathbf{k}_{\Delta_Y}[1].$$

This proves the first isomorphism in (3.1).

(i-b) For the second isomorphism, since  $F \simeq F^c$  one has

$$(Ri_! i^{-1} F)^c \simeq ((F^c)_Y)^c \simeq F \circ_X C_X \circ_X \mathbf{k}_Y \circ_X C_X.$$

Consider the map  $q: X \times \mathbb{R}^+ \times X \times \mathbb{R}^+ \times X \rightarrow X \times \mathbb{R}^+ \times X$  given by  $q(x, t, x', t', x'') = (x, tt', x'')$  and the subset

$$Q = \{(x, t, x', t', x'') : x' = tx, x' \in Y, x'' = t'x'\}$$

of the source space. Since  $Y$  is  $\mathbb{R}^+$ -simple,  $q$  induces a topological isomorphism between  $Q$  and the subset  $\{(x, t'', x'') : x, x'' \in \mathbb{R}^+ Y, x'' = t''x\}$  of the target space. Hence

$$C_X \circ_X \mathbf{k}_Y \circ_X C_X \simeq Rq_! \mathbf{k}_Q[2] \simeq R(j \times j)_! C_{\mathbb{R}^+ Y}[1],$$

where  $j: \mathbb{R}^+ Y \rightarrow X$  is the embedding. Then

$$\begin{aligned} (Ri_! i^{-1} F)^c &\simeq F \circ_X R(j \times j)_! C_{\mathbb{R}^+ Y}[1] \simeq \\ &Rj_!(F|_{\mathbb{R}^+ Y} \circ_{\mathbb{R}^+ Y} C_{\mathbb{R}^+ Y})[1] \simeq Rj_!(F|_{\mathbb{R}^+ Y})[1] \simeq F_{\mathbb{R}^+ Y}[1], \end{aligned}$$

where the third isomorphism is due to the fact that  $F|_{\mathbb{R}^+ Y}$  is conic.

(ii) For the second equivalence in the statement, it is enough to prove that for  $G \in \mathbf{D}^b(\mathbf{k}_Y)$  and  $F \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_X)$  there are isomorphisms

$$(3.2) \quad i^!(^c(Ri_! G)) \simeq G[-1], \quad ^c(Ri_* i^! F) \simeq R\Gamma_{\mathbb{R}^+ Y} F[-1].$$

These can be deduced from (3.1) by adjunction. For example, the second isomorphism follows by noticing that for any  $F' \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_X)$  one has

$$\begin{aligned} \mathrm{Hom}(F', R\Gamma_{\mathbb{R}^+ Y} F[-1]) &\simeq \mathrm{Hom}(F'_{\mathbb{R}^+ Y}[1], F) \\ &\simeq \mathrm{Hom}((Ri_! i^{-1} F')^c, F) \\ &\simeq \mathrm{Hom}(F', Ri_* i^!(^c F)) \\ &\simeq \mathrm{Hom}(F'^c, Ri_* i^! F) \\ &\simeq \mathrm{Hom}(F', ^c(Ri_* i^! F)), \end{aligned}$$

where the fourth equivalence follows from the fact that  $F$  and  $F'$  are conic.  $\square$

Let  $Y, Z$  be locally compact spaces endowed with an  $\mathbb{R}^+$ -action. Let  $X = Y \times Z$  be endowed with the diagonal  $\mathbb{R}^+$ -action. Let  $z_o \in Z$  be such that  $\mathbb{R}^+$  acts regularly on the orbit  $\mathbb{R}^+ z_o$ . Then the embedding

$$i: Y \rightarrow X, \quad y \mapsto (y, z_o)$$

identifies  $Y$  with an  $\mathbb{R}^+$ -simple closed subset of  $X$ .

**Lemma 3.5.** *For  $G \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_Y)$  there is an isomorphism*

$$(Ri_!G)^c \simeq G \boxtimes \mathbf{k}_{\mathbb{R}^+z_o}[1].$$

*Proof.* As  $G \simeq G^c$ , it is enough to prove the isomorphism in  $\mathbf{D}^b(\mathbf{k}_{Y \times Y \times Z})$

$$C_Y \circ_Y \mathbf{k}_{\Gamma_i} \circ_X C_X \simeq C_Y \boxtimes \mathbf{k}_{\mathbb{R}^+z_o}[1].$$

Denote by  $w = (t, y, y', t', y'', z, y''', z')$  a point of  $\mathbb{R}^+ \times Y \times Y \times \mathbb{R}^+ \times Y \times Z \times Y \times Z$ , and set  $Q = \{y' = ty, y' = y'', z = z_o, y''' = t'y'', z' = t'z\}$ . By (1.9) one has

$$C_Y \circ_Y \mathbf{k}_{\Gamma_i} \circ_X C_X \simeq Rq_! \mathbf{k}_Q[2],$$

for  $q(w) = (y, y''', z')$ . Set  $e(w) = (tt', y, t'y', t'^{-1}, t'y'', z, y''', z')$ . Then  $qe = q$  and  $e(Q) = \{y''' = ty, y' = y'', z = z_o, z_o = t'z'\}$ . As  $\mathbb{R}^+$  acts regularly on the orbit of  $z_o$ , one has

$$Rq_! \mathbf{k}_Q \simeq Rq_! \mathbf{k}_{e(Q)} \simeq (Rq_{23!} \mathbf{k}_{\{y'''=ty\}}) \boxtimes \mathbf{k}_{\mathbb{R}^+z_o}.$$

□

**Lemma 3.6.** *Let  $A \subset Y$  be a locally closed subset. Then there is an isomorphism*

$$(Ri_! \mathbf{k}_A)^c \simeq \mathbf{k}_{\mathbb{R}^+i(A)}[1].$$

#### 4. FOURIER-SATO TRANSFORM

Here, we recall the definition and main properties of the Fourier-Sato transform, referring to [9, §3.7] for details.

Let  $\mathbf{V}$  and  $\mathbf{V}^*$  be dual real vector spaces by the pairing

$$\mathbf{V} \times \mathbf{V}^* \rightarrow \mathbb{R}, \quad (x, y) \mapsto \langle x, y \rangle.$$

They are endowed with a natural  $\mathbb{R}^+$ -action.

**Definition 4.1.** The Fourier-Sato transform and its adjoint are the functors

$$\begin{aligned} (\cdot)^\wedge: \mathbf{D}^b(\mathbf{k}_{\mathbf{V}}) &\rightarrow \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_{\mathbf{V}^*}), & F &\mapsto F \circ_{\mathbf{V}} \mathbf{k}_{\{\langle x, y \rangle \leq 0\}}, \\ (\cdot)^\vee: \mathbf{D}^b(\mathbf{k}_{\mathbf{V}^*}) &\rightarrow \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_{\mathbf{V}}), & G &\mapsto [\mathbf{k}_{\{\langle x, y \rangle \leq 0\}}, G]_{\mathbf{V}^*}. \end{aligned}$$

One uses the same notations when interchanging the roles of  $\mathbf{V}$  and  $\mathbf{V}^*$ .

**Theorem 4.2** ([9, Theorem 3.7.9]). *The Fourier-Sato transform induces an equivalence of categories*

$$(\cdot)^\wedge: \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_{\mathbf{V}}) \xrightarrow{\sim} \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_{\mathbf{V}^*})$$

with quasi-inverse  $(\cdot)^\vee$ .

Denote by  $n$  the dimension of  $\mathbf{V}$ . For  $F \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_{\mathbf{V}})$  one has

$$(4.1) \quad F^\vee \simeq F^{\wedge a}[n],$$

where  $G^a = a^{-1}G$  for  $a$  the antipodal map  $a(y) = -y$ .

Let  $\mathbf{V}_i$  ( $i = 1, 2$ ) be a real vector space of dimension  $n_i$ . Denote by  ${}^t f: \mathbf{V}_2^* \rightarrow \mathbf{V}_1^*$  the transpose of a linear map  $f: \mathbf{V}_1 \rightarrow \mathbf{V}_2$ . For  $F_i \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_{\mathbf{V}_i})$ , one has

$$(4.2) \quad (F_1 \boxtimes F_2)^\wedge \simeq F_1^\wedge \boxtimes F_2^\wedge,$$

$$(4.3) \quad (Rf_! F_1)^\wedge \simeq {}^t f^{-1}(F_1^\wedge),$$

$$(4.4) \quad (f^{-1} F_2)^\wedge \simeq R^t f_!(F_2^\wedge)[n_2 - n_1].$$

Denote by  $s: \mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$  the vector sum  $s(x_1, x_2) = x_1 + x_2$ . The convolution of  $F, F' \in \mathbf{D}^b(\mathbf{k}_{\mathbf{V}})$  is defined by

$$F * F' = Rs_!(F \boxtimes F').$$

Following [17] (see also [6]), the right adjoint to the convolution is given by

$$\mathcal{H}om^*(F, F') = Rs_* R\mathcal{H}om(q_2^{-1} F^a, q_1^! F').$$

Noticing that the diagonal embedding is the transpose of the vector sum, for  $F, F' \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_{\mathbf{V}})$  one gets

$$(4.5) \quad (F \otimes F')^\wedge \simeq F^\wedge * F'^\wedge[n].$$

By adjunction one then has

$$(4.6) \quad \mathcal{H}om(F, F')^\vee \simeq \mathcal{H}om^*(F^\vee, F'^\vee).$$

Let us end this section by recalling some computations of Fourier transforms that we shall use later.

A subset  $\gamma \subset \mathbf{V}$  such that  $\gamma = \mathbb{R}^+ \gamma$  is called a cone. A cone  $\gamma$  is called proper if it contains no lines. Note that  $\gamma$  is convex if and only if  $\gamma + \gamma = \gamma$ . The polar of  $\gamma \subset \mathbf{V}$  is the cone

$$\gamma^\circ = \{y \in \mathbf{V}^*: \langle x, y \rangle \geq 0, \forall x \in \gamma\}.$$

**Lemma 4.3** ([9, Lemma 3.7.10]). (i) *Let  $\gamma \subset \mathbf{V}$  be a proper closed convex cone containing the origin. Then*

$$\mathbf{k}_\gamma^\wedge \simeq \mathbf{k}_{\text{Int}\gamma^\circ}.$$

(ii) *Let  $\gamma \subset \mathbf{V}$  be an open convex cone. Then*

$$\mathbf{k}_\gamma^\wedge \simeq \mathbf{k}_{\gamma^{\circ a}}[-n].$$

Let  $\mathbf{V} = \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r$ . Denote  $(x', x'', x''')$  the coordinate system on  $\mathbf{V}$  and by  $(y', y'', y''')$  the dual coordinate system on  $\mathbf{V}^*$ . Set

$$x'^2 = x_1'^2 + \cdots + x_p'^2, \quad x''^2 = x_1''^2 + \cdots + x_q''^2.$$

**Lemma 4.4** ([11, Lemma 6.2.1]). *Let*

$$\gamma = \{x'^2 - x''^2 \leq 0, x''' = 0\}$$

*be a quadratic cone. Then*

$$\mathbf{k}_\gamma^\wedge \simeq \mathbf{k}_{\{y'^2 - y''^2 \geq 0\}}[-q].$$

*Proof.* The transpose of the embedding  $i: \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbf{V}$ ,  $(x', x'') \mapsto (x', x'', 0)$ , is the projection  $p: \mathbf{V}^* \rightarrow \mathbb{R}^p \times \mathbb{R}^q$ . By (4.3), one has

$$\mathbf{k}_{\{x'^2 - x''^2 \leq 0, x''' = 0\}}^\wedge \simeq (Ri! \mathbf{k}_{\{x'^2 - x''^2 \leq 0\}})^\wedge \simeq p^{-1}(\mathbf{k}_{\{x'^2 - x''^2 \leq 0\}}^\wedge).$$

We thus reduce to the case  $r = 0$ , discussed in [11, Lemma 6.2.1].  $\square$

## 5. CONIFIED FOURIER-SATO TRANSFORM

Here, in order to apply the Fourier-Sato transform to not necessarily conic sheaves, we will compensate the lack of homogeneity by adding an extra variable.

As in the previous section, let  $\mathbf{V}$  and  $\mathbf{V}^*$  be dual real vector spaces.

Note that the conification functor on vector spaces can be expressed in terms of the Fourier-Sato transform:

**Lemma 5.1.** *For  $F \in \mathbf{D}^b(\mathbf{k}_\mathbf{V})$  one has*

$$\begin{aligned} F^c &\simeq F^{\wedge\wedge a}[n], & F^\wedge &\simeq F^{c\wedge}, \\ {}^c F &\simeq F^{\vee\vee a}[-n], & F^\vee &\simeq ({}^c F)^\vee. \end{aligned}$$

*Proof.* Since the arguments are similar, we will only discuss the first two isomorphisms.

For  $H \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_\mathbf{V})$  one has  $H \simeq H^{\vee\wedge} \simeq H^{\vee\vee a}[-n]$ . Hence there are isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}^b(\mathbf{k}_\mathbf{V})}(F, H) &\simeq \mathrm{Hom}_{\mathbf{D}^b(\mathbf{k}_\mathbf{V})}(F, H^{\vee\vee a}[-n]) \\ &\simeq \mathrm{Hom}_{\mathbf{D}^b(\mathbf{k}_\mathbf{V})}(F^{\wedge\wedge a}[n], H) \\ &\simeq \mathrm{Hom}_{\mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_\mathbf{V})}(F^{\wedge\wedge a}[n], H). \end{aligned}$$

By Proposition 2.5 (iv) and by uniqueness of the left adjoint, it follows that  $F^c \simeq F^{\wedge\wedge a}[n]$ . One then has  $F^{c\wedge} \simeq F^{\wedge\wedge a\wedge}[n] \simeq F^{\wedge\vee\wedge} \simeq F^\wedge$ .  $\square$

Consider the dual vector spaces

$$\widetilde{\mathbf{V}} = \mathbf{V} \times \mathbb{R}, \quad \widetilde{\mathbf{V}}^* = \mathbf{V}^* \times \mathbb{R}$$

by the pairing  $\langle (x, s), (y, t) \rangle = \langle x, y \rangle + st$ .

**Notation 5.2.** (i) Denote by  $\mathbf{D}_{*\{t \geq 0\}}^b(\mathbf{k}_{\widetilde{\mathbf{V}}^*})$  the full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_{\widetilde{\mathbf{V}}^*})$  whose objects  $G$  satisfy  $G * \mathbf{k}_{\{t \geq 0, y=0\}} \xrightarrow{\sim} G$ , or equivalently  $G * \mathbf{k}_{\{t > 0, y=0\}} = 0$ .

- (ii) Denote by  $\mathbf{D}_{\{t \geq 0\}*}^b(\mathbf{k}_{\tilde{V}*})$  the full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_{\tilde{V}*})$  whose objects  $G$  satisfy  $G \xrightarrow{\sim} \mathcal{H}om^*(\mathbf{k}_{\{t \geq 0, y=0\}}, G)$ , or equivalently  $\mathcal{H}om^*(\mathbf{k}_{\{t > 0, y=0\}}, G) = 0$ .

Let us identify  $V$  with an  $\mathbb{R}^+$ -simple subset of  $\tilde{V}$  by the embedding

$$i: V \rightarrow \tilde{V}, \quad x \mapsto (x, -1).$$

**Theorem 5.3.** *There are equivalences*

$$\begin{aligned} \mathbf{D}^b(\mathbf{k}_V) &\xrightarrow{\sim} \mathbf{D}_{\mathbb{R}^+, * \{t \geq 0\}}^b(\mathbf{k}_{\tilde{V}*}), & F &\mapsto (Ri_! F)^\wedge, \\ \mathbf{D}^b(\mathbf{k}_V) &\xrightarrow{\sim} \mathbf{D}_{\mathbb{R}^+, \{t \geq 0\}*}^b(\mathbf{k}_{\tilde{V}*}), & F &\mapsto (Ri_* F)^\vee, \end{aligned}$$

with quasi inverses  $G \mapsto i^{-1}(G^\vee)[-1]$  and  $G \mapsto i^!(G^\wedge)[1]$ , respectively.

The category  $\mathbf{D}_{* \{t \geq 0\}}^b(\mathbf{k}_{\tilde{V}*})$  is of the kind of categories discussed by Tamarkin in [17]. In Appendix A we show how the above functor  $F \mapsto (Ri_! F)^\wedge$  is expressed in terms of the Fourier transform considered in [17].

*Proof.* As the proofs are similar, we will only discuss the first equivalence. By Proposition 3.4, there is an equivalence

$$\mathbf{D}^b(\mathbf{k}_V) \xrightarrow{\sim} \mathbf{D}_{\mathbb{R}^+, \langle \{s < 0\} \rangle}^b(\mathbf{k}_{\tilde{V}}), \quad F \mapsto (Ri_! F)^c,$$

with quasi-inverse  $i^{-1}[-1]$ . Since  $(Ri_! F)^\wedge \simeq (Ri_! F)^{c\wedge}$ , by Theorem 4.2 we are left to prove that the Fourier-Sato transform between  $\tilde{V}$  and  $\tilde{V}^*$  induces an equivalence

$$(5.1) \quad \mathbf{D}_{\mathbb{R}^+, \langle \{s < 0\} \rangle}^b(\mathbf{k}_{\tilde{V}}) \xrightarrow{\sim} \mathbf{D}_{\mathbb{R}^+, * \{t \geq 0\}}^b(\mathbf{k}_{\tilde{V}*}).$$

By (4.2) and Lemma 4.3 (i) one has

$$\mathbf{k}_{\{s \geq 0\}}^\wedge \simeq (\mathbf{k}_V \boxtimes \mathbf{k}_{\{s \geq 0\}})^\wedge \simeq \mathbf{k}_V^\wedge \boxtimes \mathbf{k}_{\{s \geq 0\}}^\wedge \simeq \mathbf{k}_{\{t > 0, y=0\}}[-n].$$

Let  $H \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_{\tilde{V}})$ . By (4.5), one has

$$(H \otimes \mathbf{k}_{\{s \geq 0\}})^\wedge \simeq H^\wedge * \mathbf{k}_{\{t > 0, y=0\}}[1].$$

Hence the conditions  $H \otimes \mathbf{k}_{\{s \geq 0\}} = 0$  and  $H^\wedge * \mathbf{k}_{\{t > 0, y=0\}} = 0$  are equivalent.  $\square$

**Remark 5.4.** It follows from (4.1) that

$$(Ri_* F)^\vee \simeq (Ri_! F)^{\wedge a}[n+1].$$

Thus, Theorem 5.3 implies that for  $G \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_{\tilde{V}*})$  the two conditions

$$G * \mathbf{k}_{\{t > 0, y=0\}} = 0, \quad \mathcal{H}om^*(\mathbf{k}_{\{t < 0, y=0\}}, G) = 0,$$

are equivalent.

**Remark 5.5.** One can recast the equivalence (5.1) in terms of the theory of microsupport from [9]. Recall that the microsupport of  $F \in \mathbf{D}^b(\mathbf{k}_V)$  is a closed conic involutive subset  $SS(F) \subset T^*\mathbf{V}$  of the cotangent bundle. For  $A \subset T^*\mathbf{V}$ , denote by  $\mathbf{D}_{\mu A}^b(\mathbf{k}_V)$  the full subcategory of  $\mathbf{D}^b(\mathbf{k}_V)$  whose objects  $F$  satisfy  $SS(F) \subset A$ . Denote by  $\pi: T^*\mathbf{V} \rightarrow \mathbf{V}$  the projection. Since  $\text{supp}(F) = \pi(SS(F))$ , for  $S \subset \mathbf{V}$  one has  $\mathbf{D}_S^b(\mathbf{k}_V) = \mathbf{D}_{\mu \pi^{-1}(S)}^b(\mathbf{k}_V)$ .

(i) From the adjunction isomorphism

$$\text{Hom}(F \otimes \mathbf{k}_{\{s \geq 0\}}, F') \simeq \text{Hom}(F, R\Gamma_{\{s \geq 0\}}(F'))$$

one deduces that  $\mathbf{D}_{\mathbb{R}^+, \{s < 0\}}^b(\mathbf{k}_{\tilde{V}})$  is the left orthogonal to  $\mathbf{D}_{\mathbb{R}^+, \mu \{s \geq 0\}}^b(\mathbf{k}_{\tilde{V}})$ .

(ii) Note that, using  $t \in \mathbb{R}$  as coordinate, the associated symplectic coordinates in  $T^*\mathbb{R}$  are  $(t; s)$ . By Tamarkin [17],  $\mathbf{D}_{\mathbb{R}^+, * \{t \geq 0\}}^b(\mathbf{k}_{\tilde{V}^*})$  is the left orthogonal to  $\mathbf{D}_{\mathbb{R}^+, \mu \{s \leq 0\}}^b(\mathbf{k}_{\tilde{V}^*})$ .

(iii) The equivalence (5.1) then follows from [9, Theorem 5.5.5].

**Lemma 5.6.** (i) Consider the subset  $\{\langle x, y \rangle \leq t\} \subset \mathbf{V} \times \tilde{\mathbf{V}}^*$ . For  $F \in \mathbf{D}^b(\mathbf{k}_V)$  one has

$$(Ri_! F)^\wedge \simeq F \circ_{\mathbf{V}} \mathbf{k}_{\{\langle x, y \rangle \leq t\}}.$$

In particular,  $F^\wedge \simeq (Ri_! F)^\wedge|_{\mathbf{V}^* \times \{0\}}$  and  $(Ri_! F)^\wedge|_{\{y=0, t < 0\}} = 0$ .

(ii) For  $F \in \mathbf{D}_{\mathbb{R}^+}^b(\mathbf{k}_V)$  one has

$$(Ri_! F)^\wedge \simeq F^\wedge \boxtimes \mathbf{k}_{\{t \geq 0\}}$$

*Proof.* (i) is implied by the isomorphism

$$\mathbf{k}_{\Gamma_i} \circ_{\mathbf{V}} \mathbf{k}_{\{\langle x, y \rangle + st \leq 0\}} \simeq \mathbf{k}_{\{\langle x, y \rangle \leq t\}}.$$

(ii) Recall that  $(Ri_! F)^\wedge \simeq (Ri_! F)^{c\wedge}$ . By Lemma 3.5 one has

$$(Ri_! F)^c \simeq F \boxtimes \mathbf{k}_{s < 0}[1].$$

Taking the Fourier-Sato transform, the statement follows by (4.2) and Lemma 4.3 (ii).  $\square$

We end this section by computing the non homogeneous Fourier transform of  $F = \mathbf{k}_A$  for some classes of locally closed subsets  $A \subset \mathbf{V}$ . Note that, by Lemma 3.6, one has

$$(5.2) \quad (Ri_! \mathbf{k}_A)^c \simeq \mathbf{k}_{\gamma_A}[1],$$

where we denote by

$$\gamma_A = \mathbb{R}^+(i(A)) \subset \tilde{\mathbf{V}}$$

the cone generated by  $i(A)$ .

Let us first consider the case where  $A$  is a nonempty, closed, convex subset. (For the notions of support function and asymptotic cone that we now recall, see for example [1].)

The asymptotic cone of  $A$  is defined by

$$\begin{aligned}\lambda_A &= \{x \in \mathbf{V}: a + \mathbb{R}^+x \subset A, \exists a \in A\} \\ &= \{x \in \mathbf{V}: a + \mathbb{R}^+x \subset A, \forall a \in A\}.\end{aligned}$$

It is the set of directions in which  $A$  is infinite. Under the identification  $\mathbf{V} = \mathbf{V} \times \{0\} \subset \tilde{\mathbf{V}}$ , one has

$$(5.3) \quad \lambda_A = \overline{\gamma_A} \cap (\mathbf{V} \times \{0\}),$$

or equivalently  $\overline{\gamma_A} = \gamma_A \sqcup \lambda_A$ .

**Lemma 5.7.** *The cone  $\overline{\gamma_A}$  is proper if and only if  $A$  contains no affine line.*

*Proof.* It follows from (5.3) and the definition of  $\lambda_A$ , by noticing that  $\overline{\gamma_A} \subset \{s \geq 0\}$   $\square$

The support function of  $A$  is defined by

$$h_A: \mathbf{V}^* \rightarrow \mathbb{R} \cup \{+\infty\}, \quad y \mapsto \sup_{x \in A} \langle x, y \rangle.$$

It describes the signed distance from the origin of the supporting hyperplanes of  $A$ . Recall that  $h_A$  is positive homogeneous, lower semicontinuous and convex. Moreover, its effective domain (that is, the set of  $y \in \mathbf{V}^*$  such that  $h_A(y) < +\infty$ ) is  $\lambda_A^{\circ a}$ .

**Lemma 5.8.** *One has*

$$\gamma_A^\circ = \{(y, t) \in \tilde{\mathbf{V}}^*: y \in \lambda_A^\circ, t \leq -h_A(-y)\}.$$

*Proof.* By definition,

$$\gamma_A^\circ = \{(y, t) \in \tilde{\mathbf{V}}^*: t \leq \langle x, y \rangle, \forall x \in A\}.$$

It is then enough to note that  $\inf_{x \in A} \langle x, y \rangle = -h_A(-y)$  and to recall that  $-h_A(-y) = -\infty$  for  $y \notin \lambda_A^\circ$ .  $\square$

Consider the projection  $q_1: \tilde{\mathbf{V}}^* = \mathbf{V}^* \times \mathbb{R} \rightarrow \mathbf{V}^*$ .

**Lemma 5.9.** (i) *Let  $A \subset \mathbf{V}$  be a nonempty, closed, convex subset which contains no affine line. Then*

$$(Ri_! \mathbf{k}_A)^\wedge \simeq q_1^{-1} \mathbf{k}_{\text{Int} \lambda_A^\circ} \otimes \mathbf{k}_{\{t \geq -h_A(-y)\}}.$$

(ii) *Let  $A \subset \mathbf{V}$  be an nonempty, open, convex subset. Then*

$$(Ri_! \mathbf{k}_A)^\wedge \simeq q_1^{-1} \mathbf{k}_{\text{Int} \lambda_A^{\circ a}} \otimes \mathbf{k}_{\{t \geq h_A(y)\}}[-n].$$

*Proof.* (i) Note that  $\overline{\gamma_A}$  is a proper closed convex cone containing the origin and  $\gamma_A = \overline{\gamma_A} \cap \{s < 0\}$ . By (5.2) and Lemma 4.3 (i), we have

$$(Ri_! \mathbf{k}_A)^\wedge \simeq \mathbf{k}_{\gamma_A}^\wedge[1] \simeq (\mathbf{k}_{\overline{\gamma_A}} \otimes \mathbf{k}_{\{s < 0\}})^\wedge[1] \simeq \mathbf{k}_{\text{Int} \gamma_A^\circ} * \mathbf{k}_{\{t \geq 0, y=0\}}[1].$$

By Lemma 5.8,

$$\text{Int} \gamma_A^\circ = \{(y, t) \in \tilde{\mathbf{V}}^*: y \in \text{Int} \lambda_A^\circ, t < -h_A(-y)\}.$$



Then one has

$$\mathbf{k}_{\mathrm{Int}\gamma_A^\circ} * \mathbf{k}_{\{t \geq 0, y=0\}} \simeq \mathbf{k}_{\{y \in \mathrm{Int}\lambda_A^\circ, t \geq -h_A(-y)\}}[-1].$$

(ii) By (5.2), Lemma 4.3 and (4.5), we have

$$(Ri_! \mathbf{k}_A)^\wedge \simeq \mathbf{k}_{\gamma_A}^\wedge[1] \simeq \mathbf{k}_{\mathrm{Int}\gamma_A^\circ}[-n],$$

and one concludes by Lemma 5.8.  $\square$

Let us now treat a non convex case. We consider the geometric situation of Lemma 4.4, so that  $(x', x'', x''')$  is the coordinate system on  $V = \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r$ , and  $(y', y'', y''')$  is the dual coordinate system on  $V^*$ .

**Lemma 5.10.** *For  $c \geq 0$ , consider the quadric*

$$A = \{(x', x'') \in V : x'^2 - x''^2 \leq c^2, x''' = 0\}$$

and set

$$g(y) = \begin{cases} c\sqrt{y'^2 - y''^2}, & \text{for } y'^2 - y''^2 \geq 0, \\ 0, & \text{else.} \end{cases}$$

Then

$$(Ri_! \mathbf{k}_A)^\wedge \simeq q_1^{-1} \mathbf{k}_{\{y'^2 - y''^2 \geq 0\}} \otimes \mathbf{k}_{\{t \geq -g(y)\}}[-q].$$

*Proof.* For  $c = 0$  the sheaf  $\mathbf{k}_A$  is conic. The statement then follows from Lemmas 4.4 and 5.6.

For  $c > 0$  one has  $\gamma_A = \{x'^2 - x''^2 \leq c^2 s^2\} \cap \{s < 0\}$ . By (5.2), Lemma 4.4 and (4.5), it follows that

$$\begin{aligned} (Ri_! \mathbf{k}_A)^\wedge &\simeq \mathbf{k}_{\gamma_A}^\wedge[1] \simeq (\mathbf{k}_{\{x'^2 - x''^2 \leq c^2 s^2\}} \otimes \mathbf{k}_{\{s < 0\}})^\wedge[1] \\ &\simeq \mathbf{k}_{\{y'^2 - y''^2 \geq (1/c^2)t^2\}} * \mathbf{k}_{\{t \geq 0, y=0\}}[-q]. \end{aligned}$$

Since

$$\{y'^2 - y''^2 \geq (1/c^2)t^2\} = \{y'^2 - y''^2 \geq 0, |t| \leq c\sqrt{y'^2 - y''^2}\},$$

one has

$$\mathbf{k}_{\{y'^2 - y''^2 \geq (1/c^2)t^2\}} * \mathbf{k}_{\{t \geq 0, y=0\}} \simeq \mathbf{k}_{\{y'^2 - y''^2 \geq 0, t \geq -c\sqrt{y'^2 - y''^2}\}}$$

$\square$

## 6. EXPONENTIAL GROWTH: REAL CASE

Here, in order to treat functions with exponential growth, we will generalize the construction of the sheaf of tempered functions of [12] (see also [15]).

Let  $X$  be a real analytic manifold. From now on we set  $\mathbf{k} = \mathbb{C}$ . Denote by  $\mathbf{D}_{\mathbb{R}\text{-}c}^b(\mathbf{k}_X)$  the full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_X)$  whose objects have  $\mathbb{R}$ -constructible cohomology groups.

Denote by  $X_{\text{sa}}$  the subanalytic site. This is the site whose objects are open subanalytic subsets of  $X$  and whose coverings are locally finite in  $X$ . One calls subanalytic sheaf on  $X$  a sheaf on  $X_{\text{sa}}$ .

Consider the natural map

$$\rho: X \rightarrow X_{\text{sa}}.$$

The direct image functor  $\rho_*$  induces a fully faithful exact functor from  $\mathbb{R}$ -constructible sheaves to subanalytic sheaves. One thus identifies  $\mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbf{k}_X)$  as a full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_{X_{\text{sa}}})$ .

Denoting by  $\mathcal{D}b_X$  the sheaf of Schwartz's distributions, the subanalytic sheaf  $\mathcal{D}b_X^t$  of tempered distributions is defined by

$$\mathcal{D}b_X^t(U) = \mathcal{D}b_X(X)/\Gamma_{X \setminus U}(X; \mathcal{D}b_X)$$

for  $U \subset X$  an open subanalytic subset. The sheaf  $\mathcal{D}b_X^t$  is acyclic on  $X_{\text{sa}}$ .

One says that a function  $\varphi$  on  $U$  has polynomial growth at  $x_o \in X$  if it satisfies the following condition. For a local coordinate system at  $x_o$ , there exist a sufficiently small compact neighborhood  $K$  of  $x_o$  and constants  $c \geq 0$ ,  $m \in \mathbb{Z}_{>0}$  such that

$$(6.1) \quad |\varphi(x)| \leq c \left( 1 + \frac{1}{\text{dist}(K \setminus U, x)} \right)^m, \quad \forall x \in K \cap U,$$

where “dist” denotes the euclidean distance on the domain of the coordinates.

One says that  $\varphi$  has polynomial growth on  $X$  if it has polynomial growth at any  $x_o \in X$ .

One says that  $\varphi \in \mathcal{C}_X^\infty(U)$  is temperate if all of its derivatives have polynomial growth.

The subanalytic sheaf of temperate smooth functions is defined by

$$\mathcal{C}_X^{\infty, t}: U \mapsto \{\varphi \in \mathcal{C}_X^\infty(U) : \varphi \text{ is temperate}\}.$$

It is an acyclic sheaf on  $X_{\text{sa}}$ .

Note that both  $\mathcal{D}b_X^t$  and  $\mathcal{C}_X^{\infty, t}$  are  $\rho_! \mathcal{D}_X$ -modules, where  $\mathcal{D}_X$  denotes the ring of analytic finite order differential operators and  $\rho_!$  is the left adjoint to  $\rho^{-1}$ .

Recall that a function  $\delta: X \rightarrow \mathbb{R}$  is called subanalytic if its graph is a subanalytic subset of  $X \times \mathbb{R}$ . By Łojasiewicz inequalities one has

**Lemma 6.1.** *The estimate (6.1) is equivalent to*

$$|\varphi(x)| \leq c \left( 1 + \frac{1}{\delta(x)} \right)^m, \quad \forall x \in K \cap U,$$

for  $\delta \geq 0$  a subanalytic function on  $K$  with  $K \cap \partial U = \{\delta(x) = 0\}$ .

Let  $j: X \rightarrow X'$  be an open subanalytic embedding of real analytic manifolds. Denote by  $X_{j\text{-sa}}$  the site structure induced on  $X$  by  $X'_{\text{sa}}$ . This is the site whose objects are open subsets of  $X$  which are subanalytic in

$X'$  and whose coverings are locally finite in  $X'$ . Let us call  $j$ -subanalytic such open subsets.

Let us say that a sheaf  $F$  on  $X$  is  $j$ - $\mathbb{R}$ -constructible if  $Rj_!F$  is  $\mathbb{R}$ -constructible in  $X'$ . Denote by  $\mathbf{D}_{j\text{-}\mathbb{R}\text{-c}}^b(\mathbf{k}_X)$  the full triangulated category of  $\mathbf{D}^b(\mathbf{k}_X)$  whose objects have  $j$ - $\mathbb{R}$ -constructible cohomology groups. We identify  $\mathbf{D}_{j\text{-}\mathbb{R}\text{-c}}^b(\mathbf{k}_X)$  to a full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_{X_{j\text{-sa}}})$ .

The following sheaves on  $X_{j\text{-sa}}$  take into account growth conditions at infinity

$$\mathcal{D}b_{X|X'}^t = \mathcal{D}b_{X'}^t|_{X_{j\text{-sa}}}, \quad \mathcal{C}_{X|X'}^{\infty,t} = \mathcal{C}_{X'}^{\infty,t}|_{X_{j\text{-sa}}}.$$

Note that these are not sheaves of  $\rho_!\mathcal{D}_X$ -modules, but modules over the ring  $\rho_!\mathcal{D}_{X'}|_{X_{j\text{-sa}}}$ .

Set

$$\tilde{X} = X \times \mathbb{R}, \quad \tilde{X}' = X' \times \mathbf{P}(\mathbb{R}),$$

where  $\mathbf{P}(\mathbb{R})$  denotes the real projective line.

A function  $f: X \rightarrow \mathbb{R}$  is called  $j$ -subanalytic if its graph is subanalytic in  $\tilde{X}'$ . Note that by Łojasiewicz inequalities such an  $f$  has polynomial growth.

Let  $f: X \rightarrow \mathbb{R}$  be a continuous  $j$ -subanalytic function and  $U \subset X$  an open  $j$ -subanalytic subset. One says that a function  $\varphi$  on  $U$  has  $f$ -exponential growth at  $x_o \in X'$  if it satisfies the following condition. For a local coordinate system at  $x_o$ , there exist a sufficiently small compact neighborhood  $K$  of  $x_o$  and constants  $c \geq 0$ ,  $m \in \mathbb{Z}_{>0}$  such that

$$|\varphi(x)| \leq c \left( 1 + \frac{1}{\text{dist}(K \setminus U, x)} + |f(x)| \right)^m e^{f(x)}, \quad \forall x \in K \cap U.$$

Note that one gets an equivalent definition by replacing the function  $\text{dist}(K \setminus U, \cdot)$  with a subanalytic function  $\delta$  as in Lemma 6.1.

One says that  $\varphi$  has  $f$ -exponential growth on  $X'$  if it has  $f$ -exponential growth at any  $x_o \in X'$ .

One says that  $\varphi \in \mathcal{C}_X^\infty(U)$  is  $f$ -temperate if all of its derivatives have  $f$ -exponential growth.

**Definition 6.2.** The presheaf of  $f$ -temperate smooth functions on the site  $X_{j\text{-sa}}$  is defined by

$$\mathcal{C}_{X|X'}^{\infty,[f]}: U \mapsto \{\varphi \in \mathcal{C}_X^\infty(U) : \varphi \text{ is } f\text{-temperate}\}.$$

It is a presheaf of  $\rho_!\mathcal{D}_{X'}|_{X_{j\text{-sa}}}$ -modules.

Note that “ $(f+c)$ -temperate” is the same as “ $f$ -temperate” for  $c \in \mathbb{R}$ . One has

$$\mathcal{C}_{X|X'}^{\infty,t} = \mathcal{C}_{X|X'}^{\infty,[0]}.$$

Let us show how  $f$ -temperate functions are related with temperate functions with one additional variable.

Denote by  $q_1: \tilde{X} = X \times \mathbb{R} \rightarrow X$  the projection. Let  $s \in \mathbb{R}$  be the coordinate and denote by  $D_{\mathbb{R}}$  the ring of differential operators with polynomial coefficients.

**Proposition 6.3.** *Let  $f: X \rightarrow \mathbb{R}$  be a continuous  $j$ -subanalytic function. There are isomorphisms*

$$\begin{aligned} \mathcal{C}_{X|X'}^{\infty, [f]} &\simeq Rq_{1*} R\mathcal{H}om(\mathbf{k}_{\{s < -f(x)\}}, R\mathcal{H}om_{D_{\mathbb{R}}}(D_{\mathbb{R}}e^s, \mathcal{C}_{\tilde{X}|\tilde{X}'}^{\infty, t})) \\ &\simeq Rq_{1*} R\mathcal{H}om(\mathbf{k}_{\{s \geq -f(x)\}}, R\mathcal{H}om_{D_{\mathbb{R}}}(D_{\mathbb{R}}e^s, \mathcal{C}_{\tilde{X}|\tilde{X}'}^{\infty, t}))[1], \\ \mathcal{C}_{X|X'}^{\infty, t} &\simeq Rq_{1*} R\mathcal{H}om_{D_{\mathbb{R}}}(D_{\mathbb{R}}e^{is}, \mathcal{C}_{\tilde{X}|\tilde{X}'}^{\infty, t}). \end{aligned}$$

In particular, the complexes on the right hand side are concentrated in degree zero and the presheaf  $\mathcal{C}_{X|X'}^{\infty, [f]}$  is an acyclic sheaf.

*Proof.* (i) Let us prove the first isomorphism. Set

$$\mathcal{C} = Rq_{1*} R\mathcal{H}om(\mathbf{k}_{\{s < -f(x)\}}, R\mathcal{H}om_{D_{\mathbb{R}}}(D_{\mathbb{R}}e^s, \mathcal{C}_{\tilde{X}|\tilde{X}'}^{\infty, t})).$$

For  $U \subset X$  a  $j$ -subanalytic open subset, one has

$$(6.2) \quad R\Gamma(U; \mathcal{C}) \simeq (E \xrightarrow{\partial_s - 1} E),$$

where the complex on the right hand side is in degrees 0 and 1, and

$$E = \Gamma(q_1^{-1}U \cap \{s < -f(x)\}; \mathcal{C}_{\tilde{X}|\tilde{X}'}^{\infty, t}).$$

(i-a) To prove that  $\mathcal{C}$  is concentrated in degree zero, it is enough to show the surjectivity of  $\partial_s - 1$  in (6.2).

Let  $\gamma: X \rightarrow \mathbb{R}$  be a  $C^\infty$  function such that  $-2 < f + \gamma < -1$ . For  $\Psi \in E$ , a  $C^\infty$  solution  $\Phi$  to  $(\partial_s - 1)\Phi = \Psi$  is given by

$$(6.3) \quad \Phi(x, s) = e^s \int_{\gamma(x)}^s e^{-u} \Psi(x, u) du.$$

We are thus left to prove that  $\Phi \in E$ . Since the estimates for the derivatives of  $\Phi$  are similar, let us only show that  $\Phi$  has polynomial growth. Since  $\Psi$  has polynomial growth, by Lemma 6.1 any  $x_o \in X'$  has a compact neighborhood  $K$  such that there are constants  $c, m$  with

$$|\Psi(x, s)| \leq c \left( 1 + \frac{1}{\text{dist}(K \setminus U, x)} + |s| + \frac{1}{|s + f(x)|} \right)^m, \quad \forall x \in K \cap U, \forall s < -f(x).$$

Then (6.3) implies

$$|\Phi(x, s)| \leq ce^s \left| \int_{\gamma(x)}^s e^{-u} \left( 1 + \frac{1}{\text{dist}(K \setminus U, x)} + |u| + \frac{1}{|u + f(x)|} \right)^m du \right|.$$

One thus deduces that  $\Phi$  has polynomial growth from Lemma 6.4 below.

(i-b) By (6.2), an element of  $H^0\mathrm{R}\Gamma(U; \mathcal{C})$  is a solution  $\Phi \in E$  of the equation  $(\partial_s - 1)\Phi = 0$ . Thus  $\Phi(x, s) = \varphi(x)e^s$ . The map

$$\mathcal{C}_{X|X'}^{\infty, [f]}(U) \rightarrow H^0\mathrm{R}\Gamma(U; \mathcal{C}), \quad \varphi(x) \mapsto \varphi(x)e^s$$

is well defined. To show that it is an isomorphism, we have to prove that it is surjective. Given  $\Phi(x, s) = \varphi(x)e^s$  with  $\Phi \in E$ , there is an estimate

$$|\varphi(x)|e^s \leq c \left( 1 + \frac{1}{\mathrm{dist}(K \setminus U, x)} + |s| + \frac{1}{|s + f(x)|} \right)^m, \\ \forall x \in K \cap U, \forall s < -f(x).$$

Taking  $s = -f(x) - 1$ , we see that  $\varphi$  has  $f$ -exponential growth. A similar argument holds for the derivatives of  $\Phi$ .

(ii) The second isomorphism in the statement follows from the first one if one shows that

$$Rq_{1*}R\mathcal{H}om_{D_{\mathbb{R}}}(D_{\mathbb{R}}e^s, \mathcal{C}_{\tilde{X}|\tilde{X}'}^{\infty, t}) = 0.$$

This is proved in a similar way to part (i) above.

(iii) The proof of the third isomorphism in the statement is again similar to part (i) above taking  $\gamma = 0$ .  $\square$

**Lemma 6.4.** *Let  $f: X \rightarrow \mathbb{R}$  be a continuous  $j$ -subanalytic function and  $\gamma: X \rightarrow \mathbb{R}$  a  $C^\infty$  function such that  $-2 < f + \gamma < -1$ . Then, for any  $m, m' \in \mathbb{Z}_{\geq 0}$ , the function defined for  $x \in X$ ,  $s < -f(x)$  by*

$$e^s \int_{\gamma(x)}^s \frac{|u|^m e^{-u}}{|u + f(x)|^{m'}} du$$

*has polynomial growth on  $\tilde{X}'$ .*

*Proof.* Recall that  $f(x)$  has polynomial growth. By the estimate  $|u| \leq |u + f(x)| + |f(x)|$ , one reduces to prove that for  $m \in \mathbb{Z}_{\geq 0}$  the functions

$$\Phi(x, s) = e^s \int_{\gamma(x)}^s |u|^m e^{-u} du, \quad \Psi(x, s) = e^s \int_{\gamma(x)}^s \frac{e^{-u}}{|u + f(x)|^{m+1}} du,$$

have polynomial growth.

Recall that  $\int u^m e^{-u} du = P(u)e^{-u}$  for  $P$  a polynomial of degree  $m$ . Then

$$|\Phi(x, s)| \leq c(1 + |s|^m + |\gamma(x)|^m e^{s+\gamma(x)}).$$

Since  $\gamma$  has polynomial growth and  $s + \gamma(x) < 0$ , we deduce that  $\Phi$  has polynomial growth.

Since  $-2 < f + \gamma < -1$  and  $s + f(x) < 0$ , we have

$$\begin{aligned} |\Psi(x, s)| &= \left| e^{s+f(x)} \int_{\gamma(x)+f(x)}^{s+f(x)} \frac{e^{-u}}{|u|^{m+1}} du \right| \\ &\leq c \left( 1 + e^{s+f(x)} \left| \int_{-1}^{s+f(x)} \frac{e^{-u}}{|u|^{m+1}} du \right| \right). \end{aligned}$$

From the estimate  $e^{-u}/|u|^{m+1} \leq e^{-u} + 1/|u|^{m+2}$  ( $u < 0$ ), we finally get

$$|\Psi(x, s)| \leq c' \left( 1 + \frac{1}{|s + f(x)|^{m+1}} \right).$$

□

## 7. EXPONENTIAL GROWTH: COMPLEX CASE

Let  $j: X \rightarrow X'$  be an open subanalytic embedding of complex analytic manifolds. Denote by  $\mathcal{O}_X$  the sheaf of holomorphic functions and by  $\mathcal{D}_X$  the ring of holomorphic finite order differential operators.

For  $f: X \rightarrow \mathbb{R}$  a continuous  $j$ -subanalytic function, let

$$\mathcal{O}_{X|X'}^{[f]} = R\mathcal{H}om_{\rho! \mathcal{D}_{\overline{X}'}|_{X_{j\text{-sa}}}}(\rho! \mathcal{O}_{\overline{X}'}|_{X_{j\text{-sa}}}, \mathcal{C}_{X|X'}^{\infty, [f]})$$

be the Dolbeault complex of  $\mathcal{C}_{X|X'}^{\infty, [f]}$ . In particular,  $\mathcal{O}_X^t = \mathcal{O}_{X|X}^{[0]}$  is the complex of tempered holomorphic functions defined in [12]. Set

$$\mathcal{O}_{X|X'}^t = \mathcal{O}_{X|X'}^{[0]}.$$

Recall that if  $X$  is a complexification of  $M$ , then

$$(7.1) \quad \mathcal{D}b_{M|M'}^t \simeq R\mathcal{H}om(F, \mathcal{O}_{X|X'}^t),$$

where  $M'$  is the closure of  $M$  in  $X'$ ,  $F = R\mathcal{H}om(\mathbf{k}_M, \mathbf{k}_X) \simeq or_M[-n]$  and  $or_M$  is the orientation sheaf and  $n$  the dimension of  $M$ .

Denote by  $\mathbb{P}(\mathbb{C})$  the complex projective line. Set

$$\tilde{X} = X \times \mathbb{C}, \quad \tilde{X}' = X' \times \mathbb{P}(\mathbb{C})$$

and denote by  $q_1: \tilde{X} \rightarrow X$  the projection.

**Proposition 7.1.** *There are isomorphisms in  $\mathbf{D}^b(\rho! \mathcal{D}_{\overline{X}'}|_{X_{j\text{-sa}}})$*

$$\begin{aligned} \mathcal{O}_{X|X'}^{[f]} &\simeq Rq_{1*} R\mathcal{H}om(\mathbf{k}_{\{\operatorname{Re} s < -f(x)\}}, R\mathcal{H}om_{D_{\mathbb{C}}}(D_{\mathbb{C}} e^s, \mathcal{O}_{\tilde{X}|\tilde{X}'}^t)) \\ &\simeq Rq_{1*} R\mathcal{H}om(\mathbf{k}_{\{\operatorname{Re} s \geq -f(x)\}}, R\mathcal{H}om_{D_{\mathbb{C}}}(D_{\mathbb{C}} e^s, \mathcal{O}_{\tilde{X}|\tilde{X}'}^t)[1]). \end{aligned}$$

*Proof.* (i) Let us prove the first isomorphism. One has

$$\begin{aligned} \mathcal{O}_{\tilde{X}|\tilde{X}'}^t &= R\mathcal{H}om_{\rho! \mathcal{D}_{\overline{\tilde{X}'}}|_{\tilde{X}_{j\text{-sa}}}}(\rho! \mathcal{O}_{\overline{\tilde{X}'}}|_{\tilde{X}_{j\text{-sa}}}, \mathcal{C}_{\tilde{X}|\tilde{X}'}^{\infty, [f]}) \\ &\simeq R\mathcal{H}om_{\rho! \mathcal{D}_{\overline{\tilde{X}'}}|_{X_{j\text{-sa}}}}(\rho! \mathcal{O}_{\overline{\tilde{X}'}}|_{X_{j\text{-sa}}}, R\mathcal{H}om_{D_{\overline{\mathbb{C}}}}(D_{\overline{\mathbb{C}}}/\langle \bar{\partial}_s \rangle, \mathcal{C}_{\tilde{X}|\tilde{X}'}^{\infty, [f]})), \end{aligned}$$

where  $\langle \bar{\partial}_s \rangle \subset D_{\mathbb{C}}^-$  denotes the left ideal generated by  $\langle \bar{\partial}_s \rangle$ . It is then enough to prove the isomorphism

$$\mathcal{C}_{X|X'}^{\infty, [f]} \simeq Rq_{1*} R\mathcal{H}om(\mathbf{k}_{\{\operatorname{Re} s < -f(x)\}}, R\mathcal{H}om_{D_{\mathbb{C}} \boxtimes D_{\mathbb{C}}^-}(D_{\mathbb{C}} e^s \boxtimes D_{\mathbb{C}}^- / \langle \bar{\partial}_s \rangle, \mathcal{C}_{\tilde{X}|\tilde{X}'}^{\infty, t})).$$

In the identification  $\mathbb{C} = \mathbb{R} \times \mathbb{R}$  given by  $s = \lambda + i\mu$ , there is an isomorphism of  $D_{\mathbb{R} \times \mathbb{R}}$ -modules

$$D_{\mathbb{C}} e^s \boxtimes D_{\mathbb{C}}^- / D_{\mathbb{C}}^- \bar{\partial}_s \simeq D_{\mathbb{R}} e^{\lambda} \boxtimes D_{\mathbb{R}} e^{i\mu}.$$

Moreover, one has

$$\mathcal{C}_{\tilde{X}|\tilde{X}'}^{\infty, t} \simeq \mathcal{C}_{X \times \mathbb{R} \times \mathbb{R} | X \times \mathbb{P}(\mathbb{R}) \times \mathbb{P}(\mathbb{R})}^{\infty, t}.$$

The statement then follows from Proposition 6.3.

(ii) The proof of the second isomorphism is similar.  $\square$

**Remark 7.2.** It would be interesting to consider also other growth conditions, like for example those used in [13] to construct Fourier hyperfunctions.

Consider the closed embedding

$$i: X \rightarrow \tilde{X}, \quad x \mapsto (x, -1).$$

It follows from [12, Theorem 7.4.6] that one has

$$(7.2) \quad Ri_* \mathcal{O}_{X|X'}^t \simeq R\mathcal{H}om_{D_{\mathbb{C}}}(D_{\mathbb{C}} \delta(s+1), \mathcal{O}_{\tilde{X}|\tilde{X}'}^t),$$

where  $D_{\mathbb{C}} \delta(s+1) = D_{\mathbb{C}} / D_{\mathbb{C}}(s+1)$  is the  $D_{\mathbb{C}}$  module generated by the  $\delta$  function of  $s = -1$ .

## 8. LAPLACE TRANSFORM

We recall here a theorem of [11] on the Fourier-Laplace transform between temperate holomorphic functions associated with conic sheaves on dual complex vector spaces.

Let  $\mathbb{V}$  and  $\mathbb{V}^*$  be dual complex  $n$ -dimensional vector spaces by the complex pairing  $(x, y) \mapsto \langle x, y \rangle$ . Denote by  $\mathbb{P}(\mathbb{V})$  and  $\mathbb{P}(\mathbb{V}^*)$  the complex projective compactifications of  $\mathbb{V}$  and  $\mathbb{V}^*$ , respectively. Let  $j: \mathbb{V} \rightarrow \mathbb{P}(\mathbb{V})$  and  $j: \mathbb{V}^* \rightarrow \mathbb{P}(\mathbb{V}^*)$  be the embeddings.

Denote by  $D_{\mathbb{V}}$  the Weyl algebra and by

$$(\cdot)^\wedge: D_{\mathbb{V}} \rightarrow D_{\mathbb{V}^*}$$

the Fourier isomorphism. If  $(x_1, \dots, x_n)$  is a coordinate system on  $\mathbb{V}$  and  $(y_1, \dots, y_n)$  the dual coordinate system on  $\mathbb{V}^*$ , this is given by

$$x_i^\wedge = -\partial_{y_i}, \quad \partial_{x_i}^\wedge = y_i.$$

If  $N$  is a  $D_{\mathbb{V}}$ -module, denote by  $N^\wedge$  the vector space  $N$  endowed with the  $D_{\mathbb{V}^*}$ -module structure induced by  $\wedge$ .

Note that the Fourier-Sato transform between  $\mathbb{V}$  and  $\mathbb{V}^*$  is associated with the kernel  $\mathbf{k}_{\{\operatorname{Re}\langle x, y \rangle \leq 0\}}$ .

A result linking the Laplace and Fourier-Sato transform was established in [14]. This was reconsidered and generalized in [11], whose main result describes the Laplace transform of conic temperate holomorphic functions:

**Theorem 8.1** ([11, Theorem 5.2.1]). *Let  $F \in \mathbf{D}_{\mathbb{R}^+, j\text{-}\mathbb{R}\text{-c}}^b(\mathbf{k}_{\mathbb{V}})$ . The Laplace transform  $\varphi \mapsto \int \varphi(x) e^{-\langle x, y \rangle} dx$  induces an isomorphism in  $\mathbf{D}^b(D_{\mathbb{V}^*})$*

$$\operatorname{RHom}(F, \mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) \simeq \operatorname{RHom}(F^\wedge[n], \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^t).$$

In particular, for  $N \in \mathbf{D}^b(D_{\mathbb{V}^*})$  one has

$$(8.1) \quad \operatorname{RHom}(F, R\mathcal{H}om_{D_{\mathbb{V}}}(N^\wedge, \mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t)) \simeq \operatorname{RHom}(F^\wedge[n], R\mathcal{H}om_{D_{\mathbb{V}^*}}(N, \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^t)).$$

(Note that the assumption of  $N$  being quasi-good required in [11] is in fact not necessary.)

**Remark 8.2.** As shown in [16], Theorem 8.1 is reformulated in the framework of the conic subanalytic site by the isomorphism

$$(\mathcal{O}_{\mathbb{V}}^{t,c})^\wedge \simeq \mathcal{O}_{\mathbb{V}^*}^{t,c}[-n],$$

where  $\mathcal{O}_{\mathbb{V}}^{t,c}$  denotes the complex of conic tempered holomorphic functions.

## 9. CONIFIED LAPLACE TRANSFORM

Here, we extend Theorem 8.1 to sheaves which are not necessarily conic.

As in the previous section, let  $\mathbb{V}$  and  $\mathbb{V}^*$  be dual complex  $n$ -dimensional vector spaces by the complex pairing  $(x, y) \mapsto \langle x, y \rangle$ . Recall that we denote by  $j$  the embeddings  $\mathbb{V} \subset \mathbb{P}(\mathbb{V})$  and  $\mathbb{V}^* \subset \mathbb{P}(\mathbb{V}^*)$ .

Consider the dual vector spaces

$$\widetilde{\mathbb{V}} = \mathbb{V} \times \mathbb{C}, \quad \widetilde{\mathbb{V}}^* = \mathbb{V}^* \times \mathbb{C}$$

by the complex pairing  $\langle (x, s), (y, t) \rangle = \langle x, y \rangle + st$ .

In order to extend Theorem 8.1 to the case of not necessarily conic sheaves, consider the embedding

$$i: \mathbb{V} \rightarrow \widetilde{\mathbb{V}}, \quad x \mapsto (x, -1).$$

Let  $g: \mathbb{V}^* \rightarrow \mathbb{R}$  be a continuous  $j$ -subanalytic function, homogeneous of degree one. Let  $F \in \mathbf{D}_{j\text{-}\mathbb{R}\text{-c}}^b(\mathbf{k}_{\mathbb{V}})$  and  $G \in \mathbf{D}_{\mathbb{R}^+, j\text{-}\mathbb{R}\text{-c}}^b(\mathbf{k}_{\mathbb{V}^*})$ . Assume that

$$(9.1) \quad (Ri_! F)^\wedge \simeq (G \boxtimes \mathbf{k}_{\mathbb{C}}) \otimes \mathbf{k}_{\{\operatorname{Re} t \geq -g(y)\}}.$$

Note that Conjecture A.4 below suggests that this assumption is not so strong.



**Theorem 9.1.** *Assume (9.1). The Laplace transform  $\varphi \mapsto \int \varphi(x)e^{-\langle x, y \rangle} dx$  induces an isomorphism in  $\mathbf{D}^b(D_{\mathbb{V}^*})$*

$$\mathrm{RHom}(F, \mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) \simeq \mathrm{RHom}(G[n], \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^{[g]}).$$

*Proof.* By (9.1) and Proposition 7.1, one has

$$\mathrm{RHom}(G[n], \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^{[g]}) \simeq \mathrm{RHom}((Ri_!F)^\wedge[n], R\mathcal{H}om_{D_{\mathbb{C}}}(D_{\mathbb{C}}e^t, \mathcal{O}_{\tilde{\mathbb{V}}^*|\mathbb{P}(\tilde{\mathbb{V}}^*)}^t))$$

The statement then follows from the chain of isomorphisms

$$\begin{aligned} \mathrm{RHom}(F, \mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) &\simeq \mathrm{RHom}(i^{-1}((Ri_!F)^c)[-1], \mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) \\ &\simeq \mathrm{RHom}((Ri_!F)^c[-1], Ri_*\mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) \\ &\simeq \mathrm{RHom}((Ri_!F)^c[-1], R\mathcal{H}om_{D_{\mathbb{C}}}(D_{\mathbb{C}}\delta(s+1), \mathcal{O}_{\tilde{\mathbb{V}}|\mathbb{P}(\tilde{\mathbb{V}})}^t)) \\ &\simeq \mathrm{RHom}((Ri_!F)^\wedge[n], R\mathcal{H}om_{D_{\mathbb{C}}}(D_{\mathbb{C}}e^t, \mathcal{O}_{\tilde{\mathbb{V}}^*|\mathbb{P}(\tilde{\mathbb{V}}^*)}^t)). \end{aligned}$$

The first isomorphism follows from (3.1). The third one follows from (7.2). The last one follows from (8.1). In fact, since  $(s+1)^\wedge = -\partial_t + 1$ , one has  $D_{\mathbb{C}}\delta(s+1) \simeq (D_{\mathbb{C}}e^t)^\wedge$ .  $\square$

**Remark 9.2.** With notations as in Remark 8.2, in the conic subanalytic framework one has

$$(Ri_*\mathcal{O}_{\tilde{\mathbb{V}}}^{t,c})^\wedge \simeq R\mathcal{H}om_{D_{\mathbb{C}}}(D_{\mathbb{C}}e^t, \mathcal{O}_{\tilde{\mathbb{V}}^*}^{t,c})[-n-1].$$

## 10. PALEY-WIENER TYPE THEOREMS

As an application of Theorem 9.1, we obtain here some Paley-Wiener type theorems.

Recall that  $\lambda_A$  and  $h_A$  denote the asymptotic cone and the support function of a convex subset  $A \subset \mathbb{V}$ . The function  $h_A$  is continuous on  $\mathrm{Int}\lambda_A^\circ$ , and is also subanalytic if so is  $A$ .

**Corollary 10.1.** (i) *Let  $A \subset \mathbb{V}$  be a nonempty, closed, subanalytic, convex subset which contains no affine line. The Fourier-Laplace transform induces an isomorphism*

$$(10.1) \quad R\Gamma_A(\mathbb{V}; \mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t)[n] \xrightarrow{\sim} R\Gamma(\mathrm{Int}\lambda_A^\circ, \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^{[h_A(-y)]}),$$

*and these complexes are concentrated in degree zero.*

(ii) *Let  $A \subset \mathbb{V}$  be a nonempty, open, subanalytic, convex subset. The Fourier-Laplace transform induces an isomorphism*

$$(10.2) \quad R\Gamma(A; \mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) \xrightarrow{\sim} R\Gamma_{\lambda_A^\circ}(\mathbb{V}^*, \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^{[-h_A(y)]})[n],$$

*and these complexes are concentrated in degree zero.*

Note that if  $A$  is bounded, then  $\lambda_A^\circ = \mathbb{V}^*$ .

**Remark 10.2.** Here we are considering the Laplace transform with kernel  $e^{-\langle x, y \rangle}$ . For the transform with kernel  $e^{i\langle x, y \rangle}$ , one should read  $iy$  instead of  $y$  in the above statement.

**Remark 10.3.** It would be interesting to relate isomorphisms (10.1) and (10.2) with the ones induced by the Radon transform as in [4]. (For a link between Radon and Fourier transforms see [3].)

*Proof.* The fact that the complexes are concentrated in degree zero follows from [4, Theorem 5.10].

Decompose the embedding  $i: \mathbb{V} \rightarrow \widetilde{\mathbb{V}}$  as

$$\mathbb{V} \xrightarrow{i_{\mathbb{R}}} \mathbb{V} \times \mathbb{R} \xrightarrow{\ell} \mathbb{V} \times \mathbb{C} = \widetilde{\mathbb{V}},$$

where  $i_{\mathbb{R}}(x) = (x, -1)$  and  $\ell$  is induced by the embedding  $\mathbb{R} \subset \mathbb{C}$ . Note that the transpose  ${}^t\ell: \widetilde{\mathbb{V}}^* = \mathbb{V}^* \times \mathbb{C} \rightarrow \mathbb{V}^* \times \mathbb{R}$  is induced by the projection  $\mathbb{C} \rightarrow \mathbb{R}$ ,  $t \mapsto \operatorname{Re} t$ . For  $A \in \mathbb{V}$  a locally closed subset, by (4.3) and (5.2) one has

$$(10.3) \quad (Ri_! \mathbf{k}_A)^\wedge \simeq (R\ell_! Ri_{\mathbb{R}!} \mathbf{k}_A)^\wedge \simeq {}^t\ell^{-1}((Ri_{\mathbb{R}!} \mathbf{k}_A)^\wedge).$$

(i) By (10.3) and Lemma 5.9 (i) we have

$$(Ri_! \mathbf{k}_A)^\wedge \simeq (\mathbf{k}_{\operatorname{Int} \lambda_A^\circ} \boxtimes \mathbf{k}_{\mathbb{C}}) \otimes \mathbf{k}_{\{\operatorname{Re} t \geq -h_A(-y)\}}.$$

Hence (10.1) follows from Theorem 9.1.

(ii) By (10.3) and Lemma 5.9 (ii) we have

$$(Ri_! \mathbf{k}_A)^\wedge \simeq (\mathbf{k}_{\lambda_A^{\circ a}} \boxtimes \mathbf{k}_{\mathbb{C}}) \otimes \mathbf{k}_{\{\operatorname{Re} t \geq h_A(y)\}}[-2n].$$

Since  $A$  is relatively compact,  $\lambda_A^{\circ a} = \mathbb{V}^*$ . Hence (10.2) follows from Theorem 9.1.  $\square$

Let us describe some particular cases. Assume that  $\mathbb{V}$  and  $\mathbb{V}^*$  are complexifications of  $\mathbf{V}$  and  $\mathbf{V}^*$ , respectively. Denote by  $\mathbf{P}(\mathbf{V})$  and  $\mathbf{P}(\mathbf{V}^*)$  the real projective compactifications of  $\mathbf{V}$  and  $\mathbf{V}^*$ , respectively. If  $A \subset \mathbf{V}$  is a closed subanalytic subset, one has

$$\Gamma_A(\mathbf{V}; \mathcal{D}b_{\mathbf{V}|\mathbf{P}(\mathbf{V})}^t) \simeq R\Gamma_A(\mathbb{V}; \mathcal{O}_{\mathbb{V}|\mathbf{P}(\mathbf{V})}^t)[n].$$

(i) Let  $A \subset \mathbf{V}$  be a closed, convex, subanalytic, bounded subset. Then  $\lambda_A = \{0\}$  and  $h_A(-\operatorname{Re} y) = h_A(-\operatorname{Re} y) = O(|y|)$ . Thus (10.1) reads

$$\Gamma_A(\mathbf{V}; \mathcal{D}b_{\mathbf{V}}) \xrightarrow{\sim} \{\psi \in \Gamma(\mathbb{V}^*; \mathcal{O}_{\mathbb{V}^*}): \\ \exists c, \exists m, \forall y, |\psi(y)| \leq c(1 + |y|)^m e^{h_A(-\operatorname{Re} y)}\}.$$

(The estimates for the derivatives of  $\psi$  are obtained by Cauchy formula.) This is the classical Paley-Wiener theorem of [7, Theorem 7.3.1].

(ii) Let  $A \subset \mathbb{V}$  be a closed, convex, subanalytic proper cone. Then  $\lambda_A = A$  and  $h_A = 0$ . Thus (10.1) reads

$$\Gamma_A(\mathbb{V}; \mathcal{D}b_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) \xrightarrow{\sim} H^0 \mathrm{R}\Gamma(\mathrm{Int} A^\circ; \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^t).$$

Also this result is classical (see e.g. [5]).

As another application of Theorem 9.1, consider  $\mathbb{V}$  and  $\mathbb{V}^*$  as dual real vector spaces by the pairing  $(x, y) \mapsto \mathrm{Re}\langle x, y \rangle$ . Choose real coordinates  $(u) = (u', u'', u''')$  so that  $\mathbb{V} = \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r$  with  $p + q + r = 2n$ , and let  $(v) = (v', v'', v''')$  be dual real coordinates on  $\mathbb{V}^*$ .

**Corollary 10.4.** *For  $c \geq 0$ , consider the real quadric*

$$A = \{u'^2 - u''^2 \leq c^2, u''' = 0\} \subset \mathbb{V}.$$

*Then*

$$\mathrm{R}\Gamma_A(\mathbb{V}; \mathcal{O}_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) \xrightarrow{\sim} \mathrm{R}\Gamma_{\{v'^2 - v''^2 \geq 0\}}(\mathbb{V}^*, \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^{[g]})[q - n],$$

*where*

$$g(v) = \begin{cases} c\sqrt{v'^2 - v''^2}, & \text{for } v'^2 - v''^2 \geq 0, \\ 0, & \text{else.} \end{cases}$$

*Proof.* The proof goes as the one of Corollary 10.1 above using Lemma 5.10 instead of Lemma 5.9.  $\square$

Let us describe some particular cases.

(i) Let  $\mathbb{R}^p = \mathbb{V}$  and  $\mathbb{R}^q = \mathbb{R}^r = \{0\}$ . Then  $A$  is a closed ball in  $\mathbb{V}$  centered at the origin,  $g(y) = c|y| = h_A(y)$ , and we recover a particular case of (10.1).

(ii) Let  $\mathbb{V} = \mathbb{R}^p \times \mathbb{R}^q$  and  $i\mathbb{V} = \mathbb{R}^r$ . Then

$$A = \{(\mathrm{Re}x')^2 - (\mathrm{Re}x'')^2 \leq c^2\} \subset \mathbb{V},$$

and we get

$$\Gamma_A(\mathbb{V}; \mathcal{D}b_{\mathbb{V}|\mathbb{P}(\mathbb{V})}^t) \xrightarrow{\sim} H^q \mathrm{R}\Gamma_{\{(\mathrm{Re}y')^2 - (\mathrm{Re}y'')^2 \geq 0\}}(\mathbb{V}^*, \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^{[g]}).$$

Moreover,  $H^i \mathrm{R}\Gamma_{\{(\mathrm{Re}y')^2 - (\mathrm{Re}y'')^2 \geq 0\}}(\mathbb{V}^*, \mathcal{O}_{\mathbb{V}^*|\mathbb{P}(\mathbb{V}^*)}^{[g]}) = 0$  for  $i \neq q$ .

For  $q = 0$  this is the classical Paley-Wiener theorem for a closed ball in  $\mathbb{V}$  centered at the origin.

For  $c = 0$  we recover a result of Faraut-Gindikin discussed in [11, Proposition 6.2.2].

## APPENDIX A. LINK WITH TAMARKIN'S FOURIER TRANSFORM

Categories like those in Section 5 are considered by Tamarkin in [17] (see also [6]). Here, after discussing some of his constructions, we make a connection between the Fourier transform he considers and the functor discussed in Theorem 5.3. We also provide a system of generators for  $\mathbb{R}$ -constructible objects in this framework.

Let  $X$  a locally compact topological space. Denote

$$\tilde{X} = X \times \mathbb{R}$$

and let  $t \in \mathbb{R}$  be the coordinate. For  $G, G' \in \mathbf{D}^b(\mathbf{k}_{\tilde{X}})$ , set

$$(A.1) \quad G *_{\mathbb{R}} G' = Rs_!(p_1^{-1}G \otimes p_2^{-1}G'),$$

where the maps  $p_1, p_2, s: \tilde{X}^2 \rightarrow \tilde{X}$  are induced by the corresponding maps  $\mathbb{R}^2 \rightarrow \mathbb{R}$  given by the first projection, the second projection and the addition, respectively.

Note that if  $X = \mathbf{V}$  is a vector space, one has

$$G *_{\mathbb{R}} \mathbf{k}_{\{t>0\}} \simeq G * \mathbf{k}_{\{t>0, x=0\}}.$$

Thus, generalizing Notation 5.2, let  $\mathbf{D}_{*\{t \geq 0\}}^b(\mathbf{k}_{\tilde{X}})$  be the full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_{\tilde{X}})$  whose objects  $G$  satisfy  $G *_{\mathbb{R}} \mathbf{k}_{\{t>0\}} = 0$ , or equivalently  $G *_{\mathbb{R}} \mathbf{k}_{\{t \geq 0\}} \xrightarrow{\sim} G$ . Such categories are considered in [17] and we now discuss some constructions from loc. cit.

Consider the fully faithful functor

$$(A.2) \quad (\cdot)^{\sim}: \mathbf{D}^b(\mathbf{k}_X) \rightarrow \mathbf{D}^b(\mathbf{k}_{\tilde{X}}), \quad F \mapsto F \boxtimes \mathbf{k}_{\{t \geq 0\}}.$$

In particular, considering the constant sheaf  $\mathbf{k}$  on the singleton  $\{pt\}$ , one has

$$\tilde{\mathbf{k}} = \mathbf{k}_{\{t \geq 0\}}, \quad \tilde{F} = F \circ_{\{pt\}} \tilde{\mathbf{k}}.$$

Note that

$$(A.3) \quad K_{12} \circ_{X_2} \tilde{K}_{23} \simeq (K_{12} \circ_{X_2} K_{23})^{\sim}.$$

For  $f: X_1 \rightarrow X_2$ , denote by  $\tilde{f}: \tilde{X}_1 \rightarrow \tilde{X}_2$  the map  $\tilde{f} = f \times \text{id}_{\mathbb{R}}$ . Recall the notation  $X_{ij} = X_i \times X_j$ . For  $L_{ij} \in \mathbf{D}^b(\mathbf{k}_{\tilde{X}_{ij}})$  set

$$(A.4) \quad L_{12} \tilde{\circ}_{X_2} L_{23} = R\tilde{q}_{13}!(\tilde{q}_{12}^{-1}L_{12} *_{\mathbb{R}} \tilde{q}_{23}^{-1}L_{23}).$$

Note that  $G *_{\mathbb{R}} G' \simeq G \tilde{\circ}_X G'$  and that one has

$$(A.5) \quad \tilde{\mathbf{k}} \tilde{\circ}_{\{pt\}} \tilde{\mathbf{k}} \simeq \tilde{\mathbf{k}}.$$

**Proposition A.1.** *For  $K_{12} \in \mathbf{D}^b(\mathbf{k}_{X_{12}})$  and  $L_{23} \in \mathbf{D}^b(\mathbf{k}_{\widetilde{X_{23}}})$  one has*

$$\widetilde{K}_{12} \underset{X_2}{\circ} L_{23} \simeq K_{12} \underset{X_2}{\circ} (L_{23} \underset{\{pt\}}{\circ} \widetilde{\mathbf{k}}) \quad \text{in } \mathbf{D}^b(\mathbf{k}_{\widetilde{X_{13}}}).$$

*Proof.* For  $y = (x_1, x_2, x_3, t, t') \in X_1 \times X_2 \times X_3 \times \mathbb{R}^2$ , set  $p(y) = (x_1, x_2)$ ,  $q(y) = (x_2, x_3, t')$ ,  $r(y) = t$  and  $u(y) = (x_1, x_3, t + t')$ . Then both sides are isomorphic to  $Ru_!(p^{-1}K_{12} \otimes q^{-1}L_{23} \otimes r^{-1}\mathbf{k}_{\{t \geq 0\}})$ .  $\square$

**Corollary A.2.** *For  $K_{ij} \in \mathbf{D}^b(\mathbf{k}_{X_{ij}})$  one has*

$$\widetilde{K}_{12} \underset{X_2}{\circ} \widetilde{K}_{23} \simeq (K_{12} \underset{X_2}{\circ} K_{23})^\sim \quad \text{in } \mathbf{D}^b(\mathbf{k}_{\widetilde{X_{13}}}).$$

*Proof.* One has

$$\begin{aligned} \widetilde{K}_{12} \underset{X_2}{\circ} \widetilde{K}_{23} &\simeq K_{12} \underset{X_2}{\circ} (\widetilde{K}_{23} \underset{\{pt\}}{\circ} \widetilde{\mathbf{k}}) \\ &\simeq K_{12} \underset{X_2}{\circ} (K_{23} \underset{\{pt\}}{\circ} (\widetilde{\mathbf{k}} \underset{\{pt\}}{\circ} \widetilde{\mathbf{k}})) \\ &\simeq K_{12} \underset{X_2}{\circ} (K_{23} \underset{\{pt\}}{\circ} \widetilde{\mathbf{k}}) = K_{12} \underset{X_2}{\circ} \widetilde{K}_{23} \\ &\simeq (K_{12} \underset{X_2}{\circ} K_{23})^\sim. \end{aligned}$$

Where the first isomorphism follows from Proposition A.1, the third isomorphism follows from (A.5) and the last isomorphism from (A.3).  $\square$

Note that (A.4) induces a functor

$$\underset{X_2}{\circ} : \mathbf{D}_{*\{t \geq 0\}}^b(\mathbf{k}_{\widetilde{X_{12}}}) \times \mathbf{D}_{*\{t \geq 0\}}^b(\mathbf{k}_{\widetilde{X_{23}}}) \rightarrow \mathbf{D}_{*\{t \geq 0\}}^b(\mathbf{k}_{\widetilde{X_{13}}}).$$

Let  $\mathbf{V}$  and  $\mathbf{V}^*$  be dual real vector spaces by the pairing  $(x, y) \mapsto \langle x, y \rangle$ . In [17], the following analogue of the Fourier-Sato transform is considered:

$$\Phi : \mathbf{D}_{*\{t \geq 0\}}^b(\mathbf{k}_{\widetilde{\mathbf{V}}}) \rightarrow \mathbf{D}_{*\{t \geq 0\}}^b(\mathbf{k}_{\widetilde{\mathbf{V}^*}}), \quad G \mapsto G \underset{\mathbf{V}}{\circ} \mathbf{k}_{\{\langle x, y \rangle \leq t\}}.$$

Note that

$$(A.6) \quad \mathbf{k}_{\{\langle x, y \rangle \leq t\}} \underset{\{pt\}}{\circ} \widetilde{\mathbf{k}} \simeq \mathbf{k}_{\{\langle x, y \rangle \leq t\}}.$$

Recall the notations of section 5.

**Proposition A.3.** *For  $F \in \mathbf{D}^b(\mathbf{k}_{\mathbf{V}})$  one has*

$$\Phi(\widetilde{F}) \simeq (Ri_! F)^\wedge.$$

*Proof.* By Proposition A.1 and (A.6), one has

$$\Phi(\widetilde{F}) = \widetilde{F} \underset{\mathbf{V}}{\circ} \mathbf{k}_{\{\langle x, y \rangle \leq t\}} \simeq F \underset{\mathbf{V}}{\circ} (\mathbf{k}_{\{\langle x, y \rangle \leq t\}} \underset{\{pt\}}{\circ} \widetilde{\mathbf{k}}) \simeq F \underset{\mathbf{V}}{\circ} \mathbf{k}_{\{\langle x, y \rangle \leq t\}}.$$

The statement then follows from Lemma 5.6.  $\square$

Let now  $j : X \rightarrow X'$  be an open subanalytic embedding of real analytic manifolds. Let us still denote by  $j$  the embedding of  $\widetilde{X}$  in  $\widetilde{X}' = X' \times \mathbf{P}(\mathbb{R})$ .

**Conjecture A.4.** *Any  $G \in \mathbf{D}_{j\text{-}\mathbb{R}\text{-c}, *_{\{t \geq 0\}}}^b(\mathbf{k}_{\tilde{X}})$  is isomorphic to a bounded complex  $G^\bullet$  where each  $G^i$  is a direct sum, locally finite in  $\tilde{X}'$ , of sheaves of the form  $\mathbf{k}_{\{x \in U, t \geq f(x)\}}$  for  $U \subset X$  an open  $j$ -subanalytic subset and  $f: U \rightarrow \mathbb{R}$  a continuous  $j$ -subanalytic function.*

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